HOMEWORK # 7
DUE FRIDAY DECEMBER 9TH

MATH 538 FALL 2011

1. Suppose that A is a ring and that M and N are A-modules. A module L together with a short exact sequence $0 \to M \to L \to N \to 0$ is called an *extension of M and N*. For example, $M \oplus N$ is an extension of M and N with the usual short exact sequence (it is called the *trivial extensions*). We say that two extensions L and L' are equivalent if there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \sim & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & L' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Prove that there is a bijective correspondence between equivalence classes of extensions and elements of $\mathrm{Ext}^1(N, M)$. Additionally, prove that under this correspondence, the element $0 \in \mathrm{Ext}^1(N, M)$ corresponds to the trivial extension.

Solution: I don't want to write down a proof of this. Please see either:

- Theorem 3.4.3 in *Homological Algebra* by Weibel.
- Theorem 12 on page 754 of *Abstract Algebra, 2nd edition* by Dummit and Foote.
- Google.

2. Let $R = k[x, y, z]$ where k is a field. Prove that $x, y(1-x), z(1-x)$ is a regular sequence on R but $y(1-x), z(1-x), x$ is not a regular sequence on R.

Solution: Indeed, the first sequence creates module $k[y, z]$ on which $y(1-x) = y$ is a regular element and $z(1-x) = z$ is also a regular element (the two elements clearly form a regular sequence). However, reversing the order, certainly $y(1-x)$ is a regular element, but $z(1-x)$ is not a regular element on $k[x, y, z]/\langle y(1-x)\rangle$. Indeed, multiplying it by y gives us zero.

3. Suppose that $x_1, \dots, x_t \in A$ is a regular sequence on a module M. Prove that $\mathrm{Tor}_1^A(M, A/\langle x_1, \dots, x_t\rangle) = 0$.

Solution: We do this by induction on t, the length of the sequence. Then we have a short exact sequence:

$$0 \to \langle x_1\rangle \xrightarrow{\cdot x_1} A \to A/\langle x_1\rangle \to 0$$

from which we obtain the long exact sequence:

$$\mathrm{Tor}_1^A(M, A) \to \mathrm{Tor}_1^A(M, A/\langle x_1\rangle) \to \langle x_1\rangle \otimes_A M \xrightarrow{f} M$$

Now, $\mathrm{Tor}_1^A(M, A) = 0$ since A is free (and thus is its own free resolution). So it suffices to show that $\langle x_1\rangle \otimes_A M \xrightarrow{f} M$ is injective. It might be that $\langle x_1\rangle$ is not isomorphic to A since we don't know that x_1 itself is a regular element on A (just on M). However, $\langle x_1\rangle$ is still a cyclic module. Indeed, it is isomorphic to $A' = A/\mathrm{Ann}_A(x_1)$ and in particular is itself an A'-module. But x_1 is a regular element on M, so that if $z \in \mathrm{Ann}_A(x_1)$, then for any $m \in M$, $x_1(zm) = (x_1 z)m = 0$ so that $zm = 0$ as well. Thus M is naturally an A'-module. It is easy to see then that

$$\langle x_1\rangle \otimes_A M = \langle x_1\rangle \otimes_{A'} M.$$

In particular, the map $\langle x_1\rangle \otimes_A M \to M$ is identified with

$$M \cong A' \otimes_{A'} M \cong \langle x_1\rangle \otimes_{A'} M \to M$$

which is clearly just multiplication by x_1. In particular, the map f above is injective which proves that

$$\mathrm{Tor}_1^A(M, A/\langle x_1\rangle) = 0.$$

Now, the general case is similar, we have a short exact sequence:

$$0 \to \frac{\langle x_1, \dots, x_n \rangle}{\langle x_1, \dots, x_{n-1} \rangle} \to \frac{A}{\langle x_1, \dots, x_{n-1} \rangle} \to \frac{A}{\langle x_1, \dots, x_n \rangle} \to 0.$$

Tensoring with M gives us a long exact sequence

$$\mathrm{Tor}_1^A\left(M, \frac{A}{\langle x_1, \dots, x_{n-1} \rangle}\right) \to \mathrm{Tor}_1^A\left(M, \frac{A}{\langle x_1, \dots, x_{n-1} \rangle}\right) \to \left(M \otimes_A \frac{\langle x_1, \dots, x_n \rangle}{\langle x_1, \dots, x_{n-1} \rangle}\right) \to \left(M \otimes \frac{A}{\langle x_1, \dots, x_{n-1} \rangle}\right)$$

Again, now by induction, $\mathrm{Tor}_1^A\left(M, \frac{A}{\langle x_1, \dots, x_{n-1} \rangle}\right) = 0$ and so we merely need to show the injectivity of

$$\left(M \otimes_A \frac{\langle x_1, \dots, x_n \rangle}{\langle x_1, \dots, x_{n-1} \rangle}\right) \to \left(M \otimes \frac{A}{\langle x_1, \dots, x_{n-1} \rangle}\right).$$

Now, set $B = A/\langle x_1, \dots, x_{n-1} \rangle$ and $N = M \otimes_A B$, certainly

$$\left(M \otimes_A \frac{\langle x_1, \dots, x_n \rangle}{\langle x_1, \dots, x_{n-1} \rangle}\right) \cong M \otimes_A \langle x_n \rangle_B \cong (M \otimes_A B) \otimes_B \langle x_n \rangle B \cong N \otimes_B \langle x_1 \rangle_B$$

and we need to show that this injects into N. But x_n is a regular element on N, and so the argument in the base case of the induction implies the desired injection.

4. Prove that the subalgebra $S = k[u^4, u^3v, uv^3, v^4] \subseteq k[u, v]$ is not Cohen-Macaulay but that $R = k[u^4, u^3v, u^2v^2, uv^3, v^4]$ is Cohen-Macaulay.

Solution: Indeed, first we notice that in both cases, $S[u^{-4}] \cong R[u^{-4}]$ since $u^3v/u^4 = v/u$ and so $u^2v^2 = u^4(v/u)^2$. Furthermore, $S[u^{-4}] = k[u^4, u^{-4}, v/u] \cong k[a, a^{-1}, b]$ for some algebraically independent a and b. That object is a polynomial ring and easily seen to be regular (especially over an algebraically closed field, but also in general). In particular, $S[u^{-4}]$ is Cohen-Macaulay. Likewise $S[v^{-4}]$ is Cohen-Macaulay. Thus the only place which is of interest is after localizing at ideals which contain u^4 and v^4. There is only one such idea, the origin. In particular, it is harmless to localize both rings at the origin $\mathfrak{m}$. Indeed, from here on out $\mathfrak{m}$ will denote the obvious origin ideal in any polynomial ring generated by the monomials.

Now, we mod out $S_{\mathfrak{m}}$ by u^4 (which is itself a regular element since $S_{\mathfrak{m}}$ is an integral domain) and notice that clearly u^3v, uv^3 are both nilpotent. Furthermore, we notice that $(u^3v)^2$ is not zero in $S_{\mathfrak{m}}/\langle u^4 \rangle$ since it is equal to $u^4(u^2v^2)$ but u^2v^2 is not an element of $S_{\mathfrak{m}}$. However, $(u^3v)^2v^4 = (uv^3)^2u^4 = 0$ in $S_{\mathfrak{m}}/\langle u^4 \rangle$. In particular v^4 is also a zero divisor. But now consider any polynomial $f(b, d, e) \in S_{\mathfrak{m}}/\langle v^4 \rangle$ in the monomials $b = u^3v$, $d = uv^3$ and $e = v^4$. Then consider f^m for $m \gg 0$. The only way this is non-zero is if f has a λe^t term for some $\lambda \neq 0$. Then $f^m = \lambda^m e^{tm}$ (all the other terms are nilpotent). Clearly $f^m \neq 0$ in this case but then it is also a zero divisor (since it kills $b = (u^3v)^2$). Thus $f(f^{m-1}(u^3v)^2) = 0$ as well and so f is a zero divisor. We have just proven that the depth of $S_{\mathfrak{m}}/\langle u^4 \rangle$ is zero and so $S_{\mathfrak{m}}$ has depth 1.

Now, I finally claim that this ring has dimension 2. Indeed, this is easy to see since $k[u^4, v^4]_{\mathfrak{m}} \subseteq k[u^4, u^3v, uv^3, v^4]_{\mathfrak{m}}$ is clearly a finite map (since u^3v, uv^3 are certainly integral over $k[u^4, v^4]$). Note that the $\mathfrak{m}$ ideals are distinct maximal ideals. Thus $S_{\mathfrak{m}}$ has dimension 2 and so it is not Cohen-Macaulay.

Now we need to show that R is Cohen-Macaulay. Indeed, the same argument as immediately above implies that it is 2 dimensional at the origin and we already know it is Cohen-Macaulay outside of the origin by the first paragraph. Thus we merely need to show that v^4 is a regular element in $R/\langle u^4 \rangle$. Here's one approach. Consider the extension $A = k[u^4, v^4] \subseteq k[u^4, u^3v, u^2v^2, uv^3, v^4] = R$. I claim that R is a free A-module of rank 4. The basis is $\{1, u^3v, u^2v^2, uv^3\}$. It is easy to see that these elements are linearly independent over A (based on the exponents mod 4). On the other hand, they are also a spanning set (since again, all needed exponent combinations modulo 4 are obtained). But since R is a free A-module, since A is Cohen-Macaulay at the origin, so is R (any A-regular sequence becomes an R-regular sequence). This completes the proof.