HOMEWORK # 6
DUE FRIDAY NOVEMBER 18TH

MATH 538 FALL 2011

1. Let A be a ring and suppose that $\mathfrak{a}$ is an ideal. Define a ring $G_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}$ where $\mathfrak{a}^0 := A$. This is a graded ring with multiplication induced by multiplication on the Rees-algebra. If A is Noetherian, prove that $G(A)$ is also Noetherian and also that $G_{\mathfrak{a}}(A)$ is isomorphic to $G_{\hat{\mathfrak{a}}}(\hat{A})$. This ring is called the *associated graded ring*.

Solution: The generators of $\mathfrak{a}$ are elements of the degree-1 part of $G(A)$. In fact, it is easy to see that they generate $G(A)$ as an A-algebra. Now, $\mathfrak{a}$ is finitely generated since A is Noetherian, this means that $G(A)$ is a finitely generated A-algebra. By Hilbert's basis theorem, $G(A)$ is Noetherian, this proves the first part.

For the second statement, simply observe that

$$G_{\hat{\mathfrak{a}}}(\hat{A}) = \oplus_{n \geq 0} \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1} \cong \oplus_{n \geq 0} \mathfrak{a}^n/\mathfrak{a}^{n+1} = G(A).$$

2. Let A be a Noetherian ring, $\mathfrak{a} \subseteq A$ an ideal and $\hat{A}$ the $\mathfrak{a}$-adic completion. For any $x \in A$, let $\hat{x}$ denote the image of $xin\hat{A}$. Show that if x is not a zero divisor in A, then $\hat{x}$ is not a zero divisor in $\hat{A}$. However, give an example where A is an integral domain but $\hat{A}$ is not.

Solution: Consider the exact sequence:

$$0 \to A \xrightarrow{\cdot x} A$$

Tensoring with $\hat{A}$ (which is flat) yields

$$0 \to \hat{A} \xrightarrow{\cdot \hat{x}} \hat{A}$$

which proves that $\hat{x}$ is not a zero-divisor.

For the example, consider $R = k[x]$ (which is certainly a domain) completed along the ideal $\langle x(x-1) \rangle = \langle x \rangle \cap \langle x - 1 \rangle$. Now, we see (basically by the Chinese Remainder Theorem) that

$$k[x]/\langle x(x-1) \rangle^n = k[x]/\langle x^n (x-1)^n \rangle \cong k[x]/\langle x^n \rangle \oplus k[x]/\langle x-1 \rangle^n$$

In particular, it follows that $\widehat{k[x]_{\langle x \rangle \cap \langle x-1 \rangle}} \cong \widehat{k[x]_{\langle x \rangle}} \oplus \widehat{k[x]_{\langle x-1 \rangle}}$. But the right side is not a domain since it is a direct sum of two non-zero rings.

3. Let $(R, \mathfrak{m})$ be a local ring and assume that $\hat{R} = R$ (in other words, R is $\mathfrak{m}$-adically complete). For any polynomial $f \in R[x]$, let $\widetilde{f}$ denote the image of f in $(R/\mathfrak{m})[x]$. *Hensel's lemma* says the following: if $f(x)$ is monic of degree n and if there exist coprime monic polynomials $\widetilde{g}, \widetilde{h} \in (R/\mathfrak{m})[x]$ of degrees $r, n-r$ with $\widetilde{f} = \widetilde{g}\widetilde{h}$ then we can lift $\widetilde{g}, \widetilde{h}$ back to monic polynomials $g, h \in R[x]$ such that $f = gh$.

Assume Hensel's lemma without proof (or read Matsumura).

(i) Deduce from Hensel's lemma that if $\widetilde{f}$ has a root of order 1 at $\alpha \in (R/\mathfrak{m})[x]$. Then f has a root of order 1, $a \in A$ such that $\alpha = a \mod \mathfrak{m}$.

(ii) Prove that 2 is a square in the ring of 7-adic integers.

Solution: As far as I can tell, there is nothing to prove for (i). In particular, factor $\widetilde{f} = (x - \alpha)\widetilde{g}$ and then lift. Note we used the fact that $\widetilde{g}$ does NOT have a root at α (in particular, that $(x - \alpha)$ and $\widetilde{g}$ are coprime).

For (ii), we let $R = \mathbb{Z}_7$ be the 7-adic integers. Consider the element $x^2 - 2 \in R[x]$. This has a simple root $3 \in \mathbb{Z}/7[x]$. Indeed, $x^2 - 2 = (x-3)(x-4)$. Thus $x^2 - 2$ has a root of order 1 in $R[x]$ also by (i), and in particular, it has a root. That solution is the desired square root of 2.

4. [The Snake Lemma] Suppose that R is a ring and that A, B, C, D, E, F are R-modules. Suppose that:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\rho} & & \\ 0 & \longrightarrow & D & \xrightarrow[\gamma]{} & E & \xrightarrow[\delta]{} & F & \longrightarrow & 0 \end{array}$$

is a diagram where each square is commutative and the rows are exact. Set K' and C' to be the kernel and cokernel of φ. Set K and C to be the kernel and cokernel of ψ. Finally set K'' and C'' be the kernel and cokernel of ρ.

Show that there is a long exact sequence $0 \to K' \to K \to K'' \xrightarrow{d} C' \to C \to C'' \to 0$ where the maps not labeled d are induced by α, β, γ, and δ. This is not difficult, but it requires a lot of diagram chasing.

Solution: Certainly left to the reader.

5. Suppose that R is a ring and M is an R-module. A sequence of elements $x_1, \dots, x_n \in R$ is called *M-regular* if x_i is a non-zero divisor on $M/(\langle x_1, \dots x_{i-1}\rangle M)$ for each i and also if $M \neq \langle x_1, \dots, x_n\rangle M$.

Now suppose that $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is a short exact sequence of R-modules and that $x_1, \dots, x_n$ is a sequence of elements which is M'-regular and M''-regular. Prove it is M-regular also.

Solution: I will do this in a slightly more general way. First I will prove a lemma.

Lemma. *With the short exact sequence as above, suppose that $x \in R$ is a regular element on M' and M''. Then x is regular on M and furthermore, there exists a short exact sequence*

$$0 \to M'/xM' \to M/xM \to M''/xM'' \to 0$$

induced from the above sequence.

Proof. Choose $m \in M$ and suppose that $xm = 0$. It follows that $x\beta(m) = \beta(xm) = 0$ and so $\beta(m) = 0$ by the regularity of x on M''. Thus there exists $m' \in M'$ such that $\alpha(m') = m$. Then $\alpha(xm') = x\alpha(m') = xm = 0$ and so since α is injective, $xm' = 0$ which implies that $m' = 0$ by the regularity of x on M'. Thus

$$0 = \alpha(0) = \alpha(m') = m$$

which completes the proof of the first statement.

For the second, I will identify M' with its image in M. We certainly have an exact sequence:

$$M'/xM' \xrightarrow{\overline{\alpha}} M/xM \xrightarrow{\overline{\beta}} M''/xM'' \to 0$$

obtained by tensoring our original sequence with $R/\langle x\rangle$. Thus it is sufficient to show that $M'/xM' \to M/xM$ is injective. Consider an element $\overline{m'} \in M'/xM'$ (with $\overline{m'}$ corresponding to $m' \in M$) and suppose that $\overline{\alpha}(\overline{m'}) = 0$. Thus $\alpha(m') \in xM$ and so we may write $\alpha(m') = xn$ for some $n \in M$. Then, $\beta(n) \in M''$. Notice that

$$x\beta(n) = \beta(xn) = \beta(\alpha(n')) = 0$$

It follows that $\beta(n) = 0$ since x is regular on M''. Thus, by exactness, there exists $n' \in M'$ such that $\alpha(n') = n$. But then $\alpha(xn') = xn = \alpha(m')$ so that $xn' = m'$ by the injectivity of α. Thus $m' \in xM'$ and so $\overline{m'} = 0$ as desired. $\square$

Now, we apply induction and we see immediately that x_1 is regular on M, that x_2 is regular on M/x_1M, that x_3 is regular on $M/\langle x_1, x_2\rangle M$ and so on. This proves the first part of the regularity definition. For the second part, notice that

$$M/\langle x_1, \dots, x_n\rangle M \to M''/\langle x_1, \dots, x_n\rangle M''$$

is surjective by the right-exactness of tensor. But $M''/\langle x_1, \dots, x_n\rangle M''$ is non-zero by hypothesis. Thus

$$0 \neq M/\langle x_1, \dots, x_n\rangle M$$

which completes the proof.