HOMEWORK # 6
DUE FRIDAY NOVEMBER 18TH

MATH 538 FALL 2011

1. Let A be a ring and suppose that $\mathfrak{a}$ is an ideal. Define a ring $G_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}$ where $\mathfrak{a}^0 := A$. This is a graded ring with multiplication induced by multiplication on the Rees-algebra. If A is Noetherian, prove that $G(A)$ is also Noetherian and also that $G_{\mathfrak{a}}(A)$ is isomorphic to $G_{\hat{\mathfrak{a}}}(\hat{A})$. This ring is called the *associated graded ring*.

2. Let A be a Noetherian ring, $\mathfrak{a} \subseteq A$ an ideal and $\hat{A}$ the $\mathfrak{a}$-adic completion. For any $x \in A$, let $\hat{x}$ denote the image of $xin\hat{A}$. Show that if x is not a zero divisor in A, then $\hat{x}$ is not a zero divisor in $\hat{A}$. However, give an example where A is an integral domain but $\hat{A}$ is not.

3. Let $(R, \mathfrak{m})$ be a local ring and assume that $\hat{R} = R$ (in other words, R is $\mathfrak{m}$-adically complete). For any polynomial $f \in R[x]$, let $\widetilde{f}$ denote the image of f in $(R/\mathfrak{m})[x]$. *Hensel's lemma* says the following: if $f(x)$ is monic of degree n and if there exist coprime monic polynomials $\widetilde{g}$, $\widetilde{h} \in (R/\mathfrak{m})[x]$ of degrees $r, n-r$ with $\widetilde{f} = \widetilde{g}\widetilde{h}$ then we can lift $\widetilde{g}$, $\widetilde{h}$ back to monic polynomials $g, h \in R[x]$ such that $f = gh$.

Assume Hensel's lemma without proof (or read Matsumura).

(i) Deduce from Hensel's lemma that if $\widetilde{f}$ has a root of order 1 at $\alpha \in (R/\mathfrak{m})[x]$. Then f has a root of order 1, $a \in A$ such that $\alpha = a \mod \mathfrak{m}$.

(ii) Prove that 2 is a square in the ring of 7-adic integers.

4. [The Snake Lemma] Suppose that R is a ring and that A, B, C, D, E, F are R-modules. Suppose that:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \rho} & & \\ 0 & \longrightarrow & D & \xrightarrow[\gamma]{} & E & \xrightarrow[\delta]{} & F & \longrightarrow & 0 \end{array}$$

is a diagram where each square is commutative and the rows are exact. Set K' and C' to be the kernel and cokernel of φ. Set K and C to be the kernel and cokernel of ψ. Finally set K'' and C'' be the kernel and cokernel of ρ.

Show that there is a long exact sequence $0 \to K' \to K \to K'' \xrightarrow{d} C' \to C \to C'' \to 0$ where the maps not labeled d are induced by α, β, γ, and δ. This is not difficult, but it requires a lot of diagram chasing.

5. Suppose that R is a ring and M is an R-module. A sequence of elements $x_1, \ldots, x_n \in R$ is called *M-regular* if x_i is a non-zero divisor on $M/(\langle x_1, \ldots x_{i-1}\rangle M)$ for each i and also if $M \neq \langle x_1, \ldots, x_n\rangle M$.

Now suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules and that $x_1, \ldots, x_n$ is a sequence of elements which is M'-regular and M''-regular. Prove it is M-regular also.