HOMEWORK # 5
DUE FRIDAY NOVEMBER 4TH

MATH 538 FALL 2011

1. Use Nakayma's lemma and results from a worksheet to show that if $(A, \mathfrak{m})$ is a Noetherian local ring, then the maximal ideal $\mathfrak{m}$ is principal if and only if $\mathfrak{m}/\mathfrak{m}^2$ is 1-dimensional over $k = R/\mathfrak{m}$.

Solution: Clearly if $\mathfrak{m}$ is principal with generator t, then $\mathfrak{m}/\mathfrak{m}^2$ is a cyclic R-module with generator $\langle t\rangle$. But since anything in $\mathfrak{m}$ annihilates $\mathfrak{m}/\mathfrak{m}^2$, it is 1-dimensional vector space over $R/\mathfrak{m}$.

Conversely, if $\mathfrak{m}/\mathfrak{m}^2$ is 1-dimensional, then from a Corollary to Nakayama's Lemma we know that a minimal generating set for the module $M = \mathfrak{m}$ is the same size as a basis for $M/\mathfrak{m}M = \mathfrak{m}/\mathfrak{m}^2$ completing the proof.

2. First some background:

> Suppose that k is an algebraically closed field. Consider $k[\varepsilon] := k[t]/\langle t^2\rangle$. Note $\operatorname{Spec} k[\varepsilon]$ is just a single point. Thus one can think of $k[\varepsilon]$ as a point plus the data of (a single) tangent direction (which of course, if we are working over $\mathbb{C}$, is more than one real direction). Set $R = k[x_1, \dots, x_n]/I$ to be a finitely generated k-algebra and suppose we are given a surjective k-algebra map $\varphi : R \to k[\varepsilon]$. We thus have
> $$\{\text{pt}\} = \operatorname{Spec} k[\varepsilon] \to \operatorname{Spec} R$$
> so we have determined a point on $\operatorname{Spec} R$ and the map φ should also be viewed as determining a tangent direction to that point.

Now we state the problem. Let k again be an algebraically closed field and choose $f \in k[x, y]$ an non-zero non-unit element such that
$$f = g + h$$
where $g = ax + by$ is a linear polynomial (or possibly zero) and $h \in \langle x, y\rangle^2$. Set $R = k[x, y]/\langle f\rangle$. Prove that the local ring $R_{\langle x,y\rangle}$ is a DVR (discrete valuation ring) if and only if there is only one surjective map $R \to k[\varepsilon]$ which maps the unique point of $\operatorname{Spec} k[\varepsilon]$ to the the point $\langle x, y\rangle$, up to a scaling factor (you should figure out exactly what I mean, I am being purposefully vague).

Solution: Suppose that $R_{\langle x,y\rangle} = S$ is a DVR with $\mathfrak{m} = \langle x, y\rangle = \langle t\rangle$. Clearly we have a natural map $S \to S/\langle t^2\rangle = k \oplus k \cdot t \cong k[\varepsilon]$. The isomorphism is not unique though. Indeed, we can have a different isomorphism $S/\langle t^2\rangle \cong k[\varepsilon]$ which sends t to $\lambda\varepsilon$ for each $\lambda \in k \setminus 0$. This is uniqueness up to scaling (the choice of λ). On the other hand, any surjective map $S \to k[\varepsilon]$ must be of this form (since k must be sent to k, and a multiple of t must clearly be sent to a multiple of ε). It then follows from the universal property of localization that any surjective map $\varphi : R \to k[\varepsilon]$ such that $\langle x, \rangle = \varphi^{-1}(\langle\varepsilon\rangle)$ factors through the natural map $R \to S$ and so the first direction is complete.

Conversely, if R is not a DVR, then $x, y \in \langle x, y\rangle$ are both needed as generators. Then with the notation as above, we have two maps $\varphi_i : S \to k[\varepsilon]$ where $\varphi_1(x) = \varepsilon$, $\varphi_1(y) = 0$ and also $\varphi_2(x) = 0$, $\varphi_2(y) = \varepsilon$. In fact, we can also scale these maps as before. But there is no way to scale φ_1 to get φ_2 in this way.

I originally intended a third equivalence, where one can show that this happens if and only if either $a \neq 0$ or $b \neq 0$. I removed it because I thought there was already enough going on in this problem. This is pretty easy to see though, either $a \neq 0$ or $b \neq 0$ is basically the same as requiring that the map to $k[\varepsilon]$ is not the zero map.

3. Give an example of an inclusion of Noetherian rings $R \subseteq S$ such that R and S have the same Krull dimension, S is a finitely generated R-algebra, S NOT a finite R-module, and

(a) $\operatorname{Spec} S \to \operatorname{Spec} R$ is not surjective.
(b) $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective.

Solution:
(a) $R = k[x] \subseteq k[x, x^{-1}] = S$
(b) $R = k[x] \subseteq k[x, x^{-1}] \oplus k[x]/\langle x\rangle = S$ where the map on the first coordinate is the obvious inclusion and the map to the second coordinate is the canonical surjection.

4. Suppose that G is a finite group of automorphisms acting on a ring A and let A^G denote the subring of G-invariant elements (all $x \in A$ such that $\sigma(x) = x$ for all $\sigma \in G$). Prove that A is an integral extension of A^G.

Solution: Fix $x \in A$. Consider the polynomial $f(t) = \prod_{\sigma \in G}(t - \sigma(x)) \in A[t]$. Clearly $f(t)$ is monic and $f(x) = 0$ since $\mathrm{id} \in G$. We only need to show that $f(t) \in A^G[t]$. But this is easy since the coefficients of $f(t)$ are symmetric functions in $\sigma(x)$ (applying another σ will only permute the elements).

5. Suppose that R is a ring and I is an ideal. The *integral closure*[1] *of* I is the set

$$\Big\{z \in R \quad \big| \quad \text{there exists } a_1 \in I^1,\, a_2 \in I^2,\, a_3 \in I^3, \ldots, a_{n-1} \in I^{n-1}, a_n \in I^n \\ \text{such that } z^n + a_1 z^{n-1} + \cdots + a_{n-1} z^1 + a_n = 0.\Big\}$$

It is usually denoted by $\overline{I}$.

(i) Prove that $\overline{I}$ is an ideal containing I.
(ii) Prove that $\langle x^2, y^2\rangle \subseteq k[x, y]$ is not integrally closed and find its integral closure.
(iii) Prove that $\overline{(\overline{I})} = \overline{I}$.
(iv) Suppose that W is a multiplicative system, prove that $W^{-1}\overline{I} = \overline{W^{-1}I}$.

Solution:

(i) Consider the following graded ring which we introduced in our study of completion, the Rees algebra $R \oplus I \oplus I^2 \oplus I^3 \oplus \cdots = R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots = R[It]$ (the t lets me keep track of what degree of the graded ring I'm in). Consider then the element $zt \in R[It]$. To say that $z \in \overline{I}$ implies that there exist $a_1, \ldots, a_n$ as above such that

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z^1 + a_n = 0.$$

Multiplying through by t^n gives us

$$(zt)^n + a_1 t (zt)^{n-1} + \cdots + a_{n-1} t^{n-1} (zt)^1 + a_n t^n = 0.$$

In other words, it implies that $zt \in R[t]$ is integral over $R[It]$. Conversely, given any homogeneous degree one element of $R[t]$ integral over $R[It]$, one can find an equation like the one above (indeed, take whatever integral relation you construct and focus on the nth-degree). Thus zt is integral over $R[It]$ if and only if $z \in \overline{I}$. This show that $\overline{I}$ is closed under addition (since we already know that the integral closure of a ring in an over-ring is another ring).

Closure under multiplication is trivial since

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z^1 + a_n = 0$$

implies

$$(rz)^n + a_1 r (zr)^{n-1} + \cdots + a_{n-1} r^{n-1} (zr)^1 + a_n r^n = 0.$$

This completes the proof of (i) since $a_i r^i \in I^i$ (since it's an ideal).

(ii) Indeed, this ideal is not integrally closed since clearly xy is a root of $t^2 - x^2y^2$. I claim that $\langle x^2, xy, y^2\rangle$ is its integral closure. However, $\langle x^2, xy, y^2\rangle$ already contains all polynomials whose minimal-degree term has degree ≥ 2. On the other hand, for degree reasons it's clear that no polynomial with minimal-degree term

[1] This is not in agreement with Atiyah-MacDonald, but in this case, Atiyah-MacDonald is in disagreement with the literature.

of degree ≤ 1 can be in the integral closure (write down the equation and pay attention to degrees). This completes the proof of (ii).

(iii) This follows similarly to (i) using the Rees algebra idea again (integral over integral is still integral for rings).

(iv) This follows similarly to the worksheet on integral closure of rings (probably it can also be done via the Rees-algebra trick).