HOMEWORK # 5
DUE FRIDAY NOVEMBER 4TH

MATH 538 FALL 2011

1. Use Nakayma's lemma and results from a worksheet to show that if $(A, \mathfrak{m})$ is a Noetherian local ring, then the maximal ideal $\mathfrak{m}$ is principal if and only if $\mathfrak{m}/\mathfrak{m}^2$ is 1-dimensional over $k = R/\mathfrak{m}$.

2. First some background:

> Suppose that k is an algebraically closed field. Consider $k[\varepsilon] := k[t]/\langle t^2 \rangle$. Note $\operatorname{Spec} k[\varepsilon]$ is just a single point. Thus one can think of $k[\varepsilon]$ as a point plus the data of (a single) tangent direction (which of course, if we are working over $\mathbb{C}$, is more than one real direction). Set $R = k[x_1, \dots, x_n]/I$ to be a finitely generated k-algebra and suppose we are given a surjective k-algebra map $\varphi : R \to k[\varepsilon]$. We thus have
> $$\{\text{pt}\} = \operatorname{Spec} k[\varepsilon] \to \operatorname{Spec} R$$
> so we have determined a point on $\operatorname{Spec} R$ and the map φ should also be viewed as determining a tangent direction to that point.

Now we state the problem. Let k again be an algebraically closed field and choose $f \in k[x, y]$ an non-zero non-unit element such that
$$f = g + h$$
where $g = ax + by$ is a linear polynomial (or possibly zero) and $h \in \langle x, y \rangle^2$. Set $R = k[x, y]/\langle f \rangle$. Prove that the local ring $R_{\langle x,y \rangle}$ is a DVR (discrete valuation ring) if and only if there is only one surjective map $R \to k[\varepsilon]$ which maps the unique point of $\operatorname{Spec} k[\varepsilon]$ to the the point $\langle x, y \rangle$, up to a scaling factor (you should figure out exactly what I mean, I am being purposefully vague).

3. Give an example of an inclusion of Noetherian rings $R \subseteq S$ such that R and S have the same Krull dimension, S is a finitely generated R-algebra, S NOT a finite R-module, and

(a) $\operatorname{Spec} S \to \operatorname{Spec} R$ is not surjective.
(b) $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective.

4. Suppose that G is a finite group of automorphisms acting on a ring A and let A^G denote the subring of G-invariant elements (all $x \in A$ such that $\sigma(x) = x$ for all $\sigma \in G$). Prove that A is an integral extension of A^G.

5. Suppose that R is a ring and I is an ideal. The *integral closure*[1] *of* I is the set
$$\Big\{ z \in R \quad | \quad \text{there exists } a_1 \in I^1,\, a_2 \in I^2,\, a_3 \in I^3, \dots, a_{n-1} \in I^{n-1}, a_n \in I^n \text{ such that } z^n + a_1 z^{n-1} + \cdots + a_{n-1} z^1 + a_n = 0. \Big\}$$
It is usually denoted by $\overline{I}$.

(i) Prove that $\overline{I}$ is an ideal containing I.
(ii) Prove that $\langle x^2, y^2 \rangle \subseteq k[x, y]$ is not integrally closed and find its integral closure.
(iii) Prove that $\overline{(\overline{I})} = \overline{I}$.
(iv) Suppose that W is a multiplicative system, prove that $W^{-1}\overline{I} = \overline{W^{-1}I}$.

[1] This is not in agreement with Atiyah-MacDonald, but in this case, Atiyah-MacDonald is in disagreement with the literature.