HOMEWORK # 4
DUE WEDNESDAY OCTOBER 19TH

MATH 538 FALL 2011

1. Is the following true or false. If it is true, prove it. If it is false, give a counter-example. If R is a ring, M is an R-module and M_1 and M_2 are submodules of M such that $M = M_1 + M_2$, then $\mathrm{Ass}(M) = \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2)$.

Solution: This is false. Consider $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus (\mathbb{Z}/\langle 2 \rangle)$. Set $M_1 = \langle (1,0) \rangle$ and $M_2 = \langle (1,1) \rangle$. Then $\mathrm{Ass}(M) = \{0, \langle 2 \rangle\}$ but $\mathrm{Ass}(M_1) = \mathrm{Ass}(M_2) = \{0\}$.

2. Give an example of an ideal I which is not primary but which satisfies the following condition:

$$\text{If } fg \in I, \text{ then either } f^n \in I \text{ or } g^n \in I \text{ for some integer } n \gg 0.$$

Solution: The ring $k[x,y]$ and the ideal $\langle x^2, xy \rangle$. Set $f = x$ and $g = y$. It is not primary though (the same elements prove it).

3. Suppose that I and J are ideals of a Noetherian ring A. Prove that if $JA_P \subseteq IA_P$ for every $P \in \mathrm{Ass}(A/I)$, then $J \subseteq I$.

Solution: Choose $x \in J$, thus $x/1 \in I_P$ for all $P \in \mathrm{Ass}(A/I)$. Write $I = Q_1 \cap \dots \cap Q_n$ with Q_i ideals which are P_i-primary. It follows that $x/1 \in IA_{P_i} \subseteq Q_i A_{P_i}$ for all i. Thus $x \in c$ where $\rho_i : A \to A_{P_i}$ is the natural map. We proved in class (or see Reid) that $\rho_i^{-1}(Q_i A_{P_i}) = Q_i$. Thus $x \in Q_i$ for all i and the proof is complete.

4. A topological space X is called *Noetherian* if every descending chain of closed sets eventually stabilizes. Suppose that R is a Noetherian ring and prove that $\operatorname{Spec} R$ is a Noetherian topological space. However, give an example of a ring R such that $\operatorname{Spec} R$ is Noetherian but R is *NOT* Noetherian.

Solution: Closed sets of $\operatorname{Spec} R$ are in bijection with radical ideals, and so the first statement follows since a Noetherian ring can't have an infinite ascending chain of any ideals (let alone radical ideals).

For the example, consider $R = k[x, xy, xy^2, xy^3, \dots] \subseteq k[x,y]$. This ring is clearly non-Noetherian. We now analyze it's prime spectrum. Consider $x \in R$. Note that $R[x^{-1}] congk[x, x^{-1}, y]$ which is clearly Noetherian (and so has a Noetherian topological space). On the other hand, suppose that $P \in \operatorname{Spec} R$ is a prime ideal containing x. Then $(xy)^2 = (xy^2)x \in P$ and so $xy \in P$. More generally, $(xy^n)^2 = (xy^{2n})x \in P$ and so $xy^n \in P$. Thus $P = \langle x, xy, xy^2, \dots \rangle$ is maximal. In particular, $\operatorname{Spec} R$ has only one point that $\operatorname{Spec} R[x^{-1}]$ does not have. Then given any descending chain of closed subsets $Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \dots$ of $\operatorname{Spec} R$ consider the possibilities:

(1) All Z_i contain P.
(2) Z_i does not contain P for $i \geq n_0$.

In the first case, the $Z_i \cap (\operatorname{Spec} R \setminus \{P\})$ stabilizes and thus so do the Z_i's (since we may work in $\operatorname{Spec} R \setminus \{P\}$). In the second case we don't even need to intersect.

5. Suppose that R is a Noetherian ring of characteristic $p > 0$. The *Frobenius morphism on* R is the ring homomorphism $F : R \to R$ defined by the rule $F(r) = r^p$. This is a ring homomorphism because $(x+y)^p = x^p + y^p$. Suppose now that I is a ring of R. Write $I = \langle r_1, \dots, r_n \rangle$. We define $I^{[p]}$ to be the ideal $\langle r_1^p, \dots, r_n^p \rangle$.

(a) Prove that $I^{[p]}$ is independent of the choice of generators $r_1, \dots, r_n$ for I.
(b) Suppose that Q is a prime ideal of R. Is it true that $Q^{[p]}$ is Q-primary? Prove or give a counter-example.
(c) Suppose that $R = \mathbb{F}_p[x_1, \dots, x_n]$. View R a module over itself via Frobenius, and use N to denote this module (in other words, $r.x = r^p x$ for $r \in R$ and $x \in N(\cong R)$. Show that N is a free module and exhibit a basis for N over R.
(d) Suppose that I is an ideal of R. Show that $\langle F(I) \rangle = I^{[p]} \subseteq I$.

(e**) Suppose that $J \subseteq I$ are ideals of R (which you may now assume is an integral domain). Suppose that $G : I \to I$ is an additive map satisfying the rule $G(rx) = r^p G(x)$. Is it true that $G(J) \subseteq J$?

As far as I know, (e**) is an open problem. If it's true, I know of an easy (and publishable) corollary. Note that there was a typo in the definition of G. It should be $G(rx) = r^p G(x)$, *NOT* $G(r.x) = r^p G(x)$. The other question was reasonable as well though.

Solution:

(a) It is sufficient to show that if $x \in I$, then $x^p \in \langle r_1^p, \dots, r_n^p \rangle =: I^{[p]}$. But if $x \in I$ then $x = t_1 r_1 + \dots + t_n r_n$ for some t_i. But then $x^p = (t_1 r_1 + \dots + t_n r_n) = t_1^p r_1^p + \dots t_n^p r_n^p \in \langle r_1^p, \dots, r_n^p \rangle$ as desired.

(b) It is not true. For example, consider the ring $R = \mathbb{F}_3[x, y, z]/\langle xy - z^2 \rangle$ and the prime ideal $Q = \langle x, z \rangle$. Then $Q^{[3]} = \langle x^3, z^3 \rangle$. But $xzy = (xy)z = z^3 \in Q^{[3]}$. Thus if $Q^{[3]}$ was primary, either $xz \in Q^{[3]}$ or $y^n \in Q^{[3]}$ for some n. The first case is absurd by degree reasons (since then $xz = a(x, y, z)x^3 + b(x, y, z)z^3 + c(x, y, z)(xy - z^2) \in k[x, y, z]$ but the only degree two term on the right hand side are $c_0 xy$ and $c_0 z^2$ neither of which can equal xz). For the second case, if $y^n \in Q^{[3]}$, then $y^n = a(x, y, z)x^3 + b(x, y, z)z^3 + c(x, y, z)(xy - z^2) \in k[x, y, z]$ but that is also absurd since every term on the right is divisible by either x or z (and y^n is not).

(c) The basis is $\{x_1^{\lambda_1} \cdot \dots \cdot x_n^{\lambda_n} \,|\, 0 \leq \lambda_i \leq p - 1\}$. I'll let you work out the details.

(d) $F(I) = \{x^p \,|\, x \in I\}$. It is then clear that $\langle F(I) \rangle = I^{[p]}$ based upon (a).

(e) ?