HOMEWORK # 4
DUE WEDNESDAY OCTOBER 19TH

MATH 538 FALL 2011

1. Is the following true or false. If it is true, prove it. If it is false, give a counter-example. If R is a ring, M is an R-module and M_1 and M_2 are submodules of M such that $M = M_1 + M_2$, then $\mathrm{Ass}(M) = \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2)$.

2. Give an example of an ideal I which is not primary but which satisfies the following condition:

$$\text{If } fg \in I, \text{ then either } f^n \in I \text{ or } g^n \in I \text{ for some integer } n \gg 0.$$

3. Suppose that I and J are ideals of a Noetherian ring A. Prove that if $JA_P \subseteq IA_P$ for every $P \in \mathrm{Ass}(A/I)$, then $J \subseteq I$.

4. A topological space X is called *Noetherian* if every descending chain of closed sets eventually stabilizes. Suppose that R is a Noetherian ring and prove that $\operatorname{Spec} R$ is a Noetherian topological space. However, give an example of a ring R such that $\operatorname{Spec} R$ is Noetherian but R is *NOT* Noetherian.

5. Suppose that R is a Noetherian ring of characteristic $p > 0$. The *Frobenius morphism on* R is the ring homomorphism $F : R \to R$ defined by the rule $F(r) = r^p$. This is a ring homomorphism because $(x+y)^p = x^p + y^p$. Suppose now that I is a ring of R. Write $I = \langle r_1, \ldots, r_n \rangle$. We define $I^{[p]}$ to be the ideal $\langle r_1^p, \ldots, r_n^p \rangle$.

(a) Prove that $I^{[p]}$ is independent of the choice of generators $r_1, \ldots, r_n$ for I.

(b) Suppose that Q is a prime ideal of R. Is it true that $Q^{[p]}$ is Q-primary? Prove or give a counter-example.

(c) Suppose that $R = \mathbb{F}_p[x_1, \ldots, x_n]$. View R a module over itself via Frobenius, and use N to denote this module (in other words, $r.x = r^p x$ for $r \in R$ and $x \in N (\cong R)$. Show that N is a free module and exhibit a basis for N over R.

(d) Suppose that I is an ideal of R. Show that $\langle F(I) \rangle = I^{[p]} \subseteq I$.

(e**) Suppose that $J \subseteq I$ are ideals of R (which you may now assume is an integral domain). Suppose that $G : I \to I$ is an additive map satisfying the rule $G(r.x) = r^p G(x)$. Is it true that $G(J) \subseteq J$?

As far as I know, (e**) is an open problem. If it's true, I know of an easy (and publishable) corollary.