HOMEWORK # 3
DUE MONDAY OCTOBER 3RD

MATH 538 FALL 2011

1. Suppose that k is an algebraically closed field, and that R and S are two finite generated k algebras (in other words, $R = k[x_1, \dots, x_m]/I$ and $S = k[y_1, \dots, y_n]/J$. Prove that there is a natural bijection between m-Spec $R \otimes_k S$, the maximal ideals of the ring $R \otimes_k S$, with (m-Spec R) $\times$ (m-Spec S).

Hint: Consider the maps $R \to R \otimes_k S$ and $S \to R \otimes_k S$ which sends $r \mapsto r \otimes 1$ and $s \mapsto 1 \otimes s$ respectively. Use these maps to induce maps from m-Spec $R \otimes_k S$ to m-Spec R and m-Spec S respectively. Now take the product map.

Solution: Consider maps $f : R \to R \otimes_k S$ and $g : S \to R \otimes_k S$ as described in the hint. This gives us a map $(f^\# \times g^\#) : \text{m-Spec}(R \otimes_k S) \to (\text{m-Spec}\, R) \times (\text{m-Spec}\, S)$. We will call this map φ. We need to show it is bijective. We will use the letter A to denote the ring $R \otimes_k S$.

First we prove a lemma.

Lemma 0.1. *If $\mathfrak{m}$ is a maximal ideal of R and $\mathfrak{n}$ is a maximal ideal of S, then $\mathfrak{c} := \langle f(\mathfrak{m})\rangle + \langle g(\mathfrak{n})\rangle = \mathfrak{m}A + \mathfrak{n}A$ is a maximal ideal of A.*

Proof. Consider the map $f : R \to A$ and apply the functor $R/\mathfrak{m} \otimes_R \bullet$, we obtain

$$f' : R/\mathfrak{m} \to R/\mathfrak{m} \otimes_R (R \otimes_k S) \cong (R/\mathfrak{m} \otimes_R R) \otimes_k S \cong R/\mathfrak{m} \otimes_k S \cong S \cong A/(\mathfrak{m}A)$$

This map is injective because S is a free k-module (in fact every module over a vector space is free). Now consider the map $\rho \circ g : S \to A \to A/(\mathfrak{m}A)$ which is an isomorphism by above and tensor with $\bullet \otimes_S S/\mathfrak{n}$ and obtain the isomorphism

$$g'' : S/\mathfrak{n} \xrightarrow{\overline{\rho \circ g}} A/(\mathfrak{m}A) \otimes_S S/\mathfrak{n} \cong (R/\mathfrak{m} \otimes_k S/\mathfrak{n}) \cong k \cong A/(\mathfrak{m}A + \mathfrak{n}A)$$

Thus $A/(\mathfrak{m}A + \mathfrak{n}A)$ is a field and so $\mathfrak{m}A + \mathfrak{n}A$ is maximal. □

We first prove the injectivity so suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are maximal ideals of $R \otimes_k S$ and that $\varphi(\mathfrak{a}) = \varphi(\mathfrak{b})$ (so $f^{-1}(\mathfrak{a}) = f^{-1}(\mathfrak{b})$ and likewise $g^{-1}(\mathfrak{a}) = g^{-1}(\mathfrak{b})$). Consider the ideal $\langle f(f^{-1}(\mathfrak{a}))\rangle + \langle g(g^{-1}(\mathfrak{a}))\rangle = \langle f(f^{-1}(\mathfrak{b}))\rangle + \langle g(g^{-1}(\mathfrak{b}))\rangle$. This is a maximal ideal, by the Lemma, contained inside both $\mathfrak{a}$ and $\mathfrak{b}$ and so the injectivity of φ is done.

Now we prove the surjectivity of φ. But this is easy since given $\mathfrak{m}$ and $\mathfrak{n}$ and constructing $\mathfrak{c}$ as in the lemma, it is clear that $f^{-1}(\mathfrak{c}) \supseteq \mathfrak{m}$ (and so we must have equality) and likewise for $\mathfrak{n}$.

2. Show that problem 1. is false if k is not algebraically closed. Also show that it doesn't hold if m-Spec is replaced by Spec (even if k is algebraically closed).

Solution: We used the fact that k was algebraically closed, and so $R/\mathfrak{m} \cong k$ when we wrote $R/\mathfrak{m} \otimes_k S \cong S$ and also $(R/\mathfrak{m} \otimes_k S/\mathfrak{n}) \cong k$. Consider the $k = \mathbb{R}$ and $R \cong S \cong \mathbb{C}$. Then $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has 2 prime ideals and not one, they are $\langle i \otimes 1 - 1 \otimes i\rangle$ and $\langle 1 \otimes i - i \otimes 1$.

For the second part, set $R = k[x]$ and $S = k[y]$. Then it is easy to see that $R \otimes_k S \cong k[x, y]$. The prime ideal $Q = \langle x - y\rangle$ satisfies $\varphi Q = (\langle 0\rangle, \langle 0\rangle)$ which is the same as $\varphi(0)$.

3. Suppose that R is a ring and M_i are R-modules. Prove that each of the M_i is flat if and only if the direct sum $\bigoplus_i M_i$ is flat.

Hint: Recall that a module is flat if the functor $\cdot \mapsto \cdot \otimes_R M$ is an exact functor.

Solution: This follows immediately from the fact that tensor commutes with direct sum (which I will not check).

4. In class we sketched the proof of Hom-Tensor adjointness. In other words, that if R is a ring and L, M, N are R-modules, then

$$\mathrm{Hom}_R(L \otimes_R M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N)).$$

In class, we only partially verified that there was a bijection of sets. Carefully write down a proof that they are isomorphic paying particular attention to verifying that these are isomorphic as *R-modules*. Additionally, prove that the isomorphism is *functorial* in the L-term (it's functorial in the other terms too, but let's just do one). This means that if $f : L_1 \to L_2$ is an R-module homomorphism, prove that there is a commutative diagram (in other words, it doesn't matter how you traverse it)

$$\begin{array}{ccc} \mathrm{Hom}_R(L_1 \otimes_R M, N) & \longleftarrow & \mathrm{Hom}_R(L_2 \otimes_R M, N) \\ \sim\updownarrow & & \updownarrow\sim \\ \mathrm{Hom}_R(L_1, \mathrm{Hom}_R(M, N)) & \longleftarrow & \mathrm{Hom}_R(L_2, \mathrm{Hom}_R(M, N)) \end{array}$$

where the horizontal arrows are induced by f (I'll leave it to you exactly how to induce the maps).

Solution: I refuse to write down any such homological algebra carefully, as it is impossible to read. This is something that must be done quietly in alone in a preferably dark room. I will also not be grading this problem. However, if you are desperate, this is I'm sure done carefully in many algebra texts. Simply search for "Hom-Tensor adjointness".

5. Prove that $\mathrm{Hom}_R(\cdot, P)$ is left exact for any R-module P (in other words, if $0 \to Q' \xrightarrow{\alpha} Q \xrightarrow{\beta} Q'' \to 0$ is exact, then $0 \to \mathrm{Hom}_R(Q'', P) \xrightarrow{\beta^\vee} \mathrm{Hom}_R(Q, P) \xrightarrow{\alpha^\vee} \mathrm{Hom}_R(Q', P)$ is also exact).

Solution: Choose $\varphi \in \mathrm{Hom}_R(Q'', P)$ and suppose that the composition $Q \xrightarrow{\beta} Q'' \xrightarrow{\varphi} P$ is zero, but then since β is surjective, φ is also zero, this shows the exactness in the middle.

Certainly also if $\varphi : Q'' \to P$ induces $Q \xrightarrow{\beta} Q'' \xrightarrow{\varphi} P$ as above (ie, $\varphi \circ \beta \in \mathrm{Image}(\beta^\vee)$), then the composition $Q' \xrightarrow{\alpha} Q \xrightarrow{\beta} Q'' \to P$ is zero since $Q' \to Q \to Q''$ is already zero. Thus $\mathrm{Image}(\beta^\vee) \subseteq \ker(\alpha^\vee)$.

Suppose now that $\psi : Q \to P$ is such that $Q' \xrightarrow{\alpha} Q \xrightarrow{\psi} P$ is zero (in other words, $\psi \in \ker(\alpha^\vee)$). Thus we have a factorization $Q \to Q/Q' \to P$ induced as in the first isomorphism theorem. But $Q/Q' \cong Q''$ and so $\psi \in \mathrm{Image}(\beta^\vee)$ as well.

6. Show that if $Q' \xrightarrow{\alpha} Q \xrightarrow{\beta} Q''$ is a sequence of maps (not necessarily exact) and that $\mathrm{Hom}_R(Q'', P) \xrightarrow{\beta^\vee} \mathrm{Hom}_R(Q, P) \xrightarrow{\alpha^\vee} \mathrm{Hom}_R(Q', P)$ is exact for all R-modules P, then $Q' \to Q \to Q''$ is also exact. This is a very special case of something called Yoneda's lemma.

Solution: Indeed, first we need to show that $\beta \circ \alpha = 0$. Take $P = Q''$ and so we know that $\mathrm{Hom}_R(Q'', Q'') \xrightarrow{\beta^\vee} \mathrm{Hom}_R(Q, Q'') \xrightarrow{\alpha^\vee} \mathrm{Hom}_R(Q', Q'')$ is exact. This means in particular that $\alpha^\vee(\beta^\vee(\mathrm{id}_{Q''})) = 0$. But this just means that the composition

$$Q' \xrightarrow{\alpha} Q \xrightarrow{\beta} Q'' \xrightarrow{\mathrm{id}_{Q''}} Q''$$

is zero which clearly implies that $\beta \circ \alpha = 0$.

For the second part, we need to show that $\ker \beta \subseteq \mathrm{Image}\, \alpha$. This time take $P = Q/\mathrm{Image}(Q') = Q/\alpha(Q')$ and consider $\phi \in \mathrm{Hom}_R(Q, P) = \mathrm{Hom}_R(Q, Q/\alpha(Q'))$ to be the natural surjection. Certainly the composition $\alpha^\vee(\phi)$ which is simply the composition

$$Q' \xrightarrow{\alpha} Q \xrightarrow{\phi} P$$

is zero by the above work. Thus $\phi = \beta^\vee(\psi)$ for some $\psi \in \mathrm{Hom}_R(Q'', P)$. In other words, the following composition

$$Q \xrightarrow{\beta} Q'' \xrightarrow{\psi} P$$

is ϕ. But then $\alpha Q' = \ker \phi \subseteq \ker \beta$ and the proof is complete.

7. Combine the previous three problems to prove the that $\otimes M$ is right exact for any R-module M.

Hint: First fix $0 \to L' \to L \to L'' \to 0$ to be a short exact sequence. Use problem 6. to conclude that it is enough to show that $0 \to \mathrm{Hom}_R(L'' \otimes M, N) \to \mathrm{Hom}_R(L \otimes M, N) \to \mathrm{Hom}_R(L' \otimes M, N)$ is exact. Now use problem 4. followed immediately by problem 5. to complete the proof.

Solution: Using the notation from the hint, we use problem 6 in the way described. However, by problem **4**,

$$0 \to \mathrm{Hom}_R(L'' \otimes M, N) \to \mathrm{Hom}_R(L \otimes M, N) \to \mathrm{Hom}_R(L' \otimes M, N)$$

is exact if and only if

$$0 \to \mathrm{Hom}_R(L'', \mathrm{Hom}_R(M, N)) \to \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(L', \mathrm{Hom}_R(M, N)$$

is exact. This latter is exact by problem 5. and the proof is completed.