HOMEWORK # 3
DUE MONDAY OCTOBER 3RD

MATH 538 FALL 2011

1. Suppose that k is an algebraically closed field, and that R and S are two finite generated k algebras (in other words, $R = k[x_1, \dots, x_m]/I$ and $S = k[y_1, \dots, y_n]/J$. Prove that there is a natural bijection between m-Spec $R \otimes_k S$, the maximal ideals of the ring $R \otimes_k S$, with (m-Spec R) $\times$ (m-Spec S).

Hint: Consider the maps $R \to R \otimes_k S$ and $S \to R \otimes_k S$ which sends $r \mapsto r \otimes 1$ and $s \mapsto 1 \otimes s$ respectively. Use these maps to induce maps from m-Spec $R \otimes_k S$ to m-Spec R and m-Spec S respectively. Now take the product map.

2. Show that problem 1. is false if k is not algebraically closed. Also show that it doesn't hold if m-Spec is replaced by Spec (even if k is algebraically closed).

3. Suppose that R is a ring and M_i are R-modules. Prove that each of the M_i is flat if and only if the direct sum $\bigoplus_i M_i$ is flat.

Hint: Recall that a module is flat if the functor $\cdot \mapsto \cdot \otimes_R M$ is an exact functor.

4. In class we sketched the proof of Hom-Tensor adjointness. In other words, that if R is a ring and L, M, N are R-modules, then

$$\mathrm{Hom}_R(L \otimes_R M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N)).$$

In class, we only partially verified that there was a bijection of sets. Carefully write down a proof that they are isomorphic paying particular attention to verifying that these are isomorphic as *R-modules*. Additionally, prove that the isomorphism is *functorial* in the L-term (it's functorial in the other terms too, but let's just do one). This means that if $f : L_1 \to L_2$ is an R-module homomorphism, prove that there is a commutative diagram (in other words, it doesn't matter how you traverse it)

$$\begin{array}{ccc}
\mathrm{Hom}_R(L_1 \otimes_R M, N) & \longleftarrow & \mathrm{Hom}_R(L_2 \otimes_R M, N) \\
\sim\updownarrow & & \updownarrow\sim \\
\mathrm{Hom}_R(L_1, \mathrm{Hom}_R(M, N)) & \longleftarrow & \mathrm{Hom}_R(L_2, \mathrm{Hom}_R(M, N))
\end{array}$$

where the horizontal arrows are induced by f (I'll leave it to you exactly how to induce the maps).

5. Prove that $\mathrm{Hom}_R(\cdot, P)$ is left exact for any R-module P (in other words, if $0 \to Q' \to Q \to Q'' \to 0$ is exact, then $0 \to \mathrm{Hom}_R(Q'', P) \to \mathrm{Hom}_R(Q, P) \to \mathrm{Hom}_R(Q', P)$ is also exact).

6. Show that if $Q' \to Q \to Q''$ is a sequence of maps (not necessarily exact) and that $\mathrm{Hom}_R(Q'', P) \to \mathrm{Hom}_R(Q, P) \to \mathrm{Hom}_R(Q', P)$ is exact for all R-modules P, then $Q' \to Q \to Q''$ is also exact. This is a very special case of something called Yoneda's lemma.

7. Combine the previous three problems to prove the that $\otimes M$ is right exact for any R-module M.

Hint: First fix $0 \to L' \to L \to L'' \to 0$ to be a short exact sequence. Use problem 6. to conclude that it is enough to show that $0 \to \mathrm{Hom}_R(L'' \otimes M, N) \to \mathrm{Hom}_R(L \otimes M, N) \to \mathrm{Hom}_R(L' \otimes M, N)$ is exact. Now use problem 4. followed immediately by problem 5. to complete the proof.