HOMEWORK # 2
DUE FRIDAY SEPTEMBER 16TH

MATH 538 FALL 2011

1. Suppose that $R := \mathbb{Z}/\langle m \rangle$ and that $S^{-1}\{1, n, n^2, n^3, \dots\}$ is a multiplicative system. Determine $S^{-1}R$.
Hint: Note $S^{-1}R \cong (S^{-1}\mathbb{Z})/(S^{-1}\langle m \rangle)$ (we will prove this in class soon, or see the book).

2. Suppose that $f : R \to S$ is a ring homomorphism and $f^{\#} : \operatorname{Spec} S \to \operatorname{Spec} R$ is the induced continuous map on topological spaces.

(a) Suppose that f is surjective, prove that $f^{\#}$ is injective.
(b) Suppose that $f^{\#}$ is injective and give an example to show that f need not be surjective.
(c) Suppose that $f : R \to S$ is injective, give an example to show that $f^{\#}$ need not be surjective, but instead that the image of $f^{\#}$ is always dense.
(d) Consider the converse to (c) (does $f^{\#}$ having dense image imply that f is injective), is it true? If so prove it. If not, can you find conditions on the rings which imply that it is true?

3. Suppose that k is a field and that $f : R \to S$ is a map between finitely generated k-algebras (this means that R is of the form $k[x_1, \dots, x_n]/I$, and likewise with S, and also that f sends k to k). Show that the function $f^{\#} : \operatorname{Spec} S \to \operatorname{Spec} R$ sends maximal ideals to maximal ideals.
Hint: You may use the fact that if K is a field and $L \supseteq K$ a field extension such that L is a finitely generated K-algebra, then L is a finite field extension. You may also use that an integral domain which is a finite extension of a field is itself a field.

4. Suppose that A is a ring and that for each prime ideal $P \in \operatorname{Spec} A$, the local ring $A_P := (A \setminus P)^{-1}A$ is an integral domain. Show that A need not be an integral domain.

However, if M and N are A-modules with a map $f : M \to N$. Consider the induced map $f_P : (A \setminus P)^{-1}M =: M_P \to N_P := (A \setminus P)^{-1}N$ for each $P \in \operatorname{Spec} A$. Show that f is surjective (respectively injective) if and only if f_P is surjective (respectively injective) for every $P \in \operatorname{Spec} A$.

5. Suppose that R and S are rings. Consider the ring $R \oplus S$ and compare its prime spectrum to that of R and S individually. Use this to give a geometric (hand-wavy) explanation for the fact that there isn't a natural ring homomorphism $R \to R \oplus S$.

6.* Suppose that R and S are rings, I is an ideal of R with canonical projection $f : R \to R/I$. Further suppose that we are given a ring homomorphism $g : S \to R/I$. Consider the following set.

$$C = \{(r, s) \in R \oplus S | f(r) = g(s)\}.$$

(a) Show that C is a subring of $R \oplus S$. Note that C has natural maps to R and to S (call them p_1 and p_2 respectively).
(b) Show that the elements of $\operatorname{Spec} C$ are in bijection with the set

$$\Big((\operatorname{Spec} R) \setminus V(I) \Big) \coprod (\operatorname{Spec} S).$$

Hint: Consider a prime in $\operatorname{Spec} C$. There are two possibilities, it either contains $p_1^{-1}(I)$ or it does not. In the latter case, invert an appropriate element and analyze what happens.

(c) Describe geometrically $\operatorname{Spec} C$ in the following examples where k is an algebraically closed field:
  (i) $R := k[x]$, $I = \langle (x^2 - 1) \rangle$ and $S = k$.
  (ii) $R := k[x, y]$, $I = \langle x \rangle$ and $S = k$.

The following philosophical statement on the elements of C might help with this problem. C is made up of the functions in $R \oplus S$ that agree on the set $V(I)$.