WORKSHEET #3 – MATH 536
SPRING

DUE MONDAY, FEBRUARY 24TH

Let H and K be groups and let $\phi : K \longrightarrow \mathrm{Aut}(H)$ be a group homomorphism. We get a left action of K on H by $k.h = (\phi(k))(h)$. Let

$$G = H \times K$$

with the following binary operation:

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1.h_2), k_1k_2).$$

This is called the *semi-direct product of K and H*, and is denoted by $H \rtimes K$.

1. Show that $G = H \rtimes K$ is a group.

Solution: We have a binary operation and so we check the various properties. Given $k \in K$, I will use $\phi_k : H \longrightarrow H$ to denote the corresponding homomorphism.

Associativity:

$$\big((h_1, k_1)(h_2, k_2)\big)(h_3, k_3) = (h_1(k_1.h_2), k_1k_2)(h_3, k_3) = (h_1(k_1.h_2)((k_1k_2).h3), k_1k_2k_3)$$

and

$$(h_1, k_1)\big((h_2, k_2)(h_3, k_3)\big) = (h_1, k_1)(h_2(k_2.h_3), k_2k_3) = (h_1(k_1.(h_2(k_2.h3))), k_1k_2k_3)$$

. Note

$$h_1(k_1.(h_2(k_2.h3))) = h_1\phi_{k_1}(h_2\phi_{k_2}(h_3)) = h_1\phi_{k_1}(h_2)\phi_{k_1}(\phi_{k_2}(h_3)) = h_1(k_1.h_2)\phi_{k_1k_2}(h_3) = h_1(k_1.h_2)(k_1k_2).(h_3)$$

as desired.

Identity: The element $(1_H, 1_K)$ is the identity since

$$(1_H, 1_K)(h_2, k_2) = (1_H(1_K.h_2), 1_Kk_2) = (h_2, k_2)$$

and

$$(h_1, k_1)(1_H, 1_K) = (h_1(k_1.1_H), k_11_K) = (h_1, k_1)$$

note that $k_1.1_H$ is really $\phi_{k_1}(1_H) = 1_H$.

Inverses: Given (h, k) set $x = \phi_k^{-1}(h^{-1})$ and consider (x, k^{-1}) so that

$$(h, k)(x, k^{-1}) = (h(k.x), kk^{-1}) = (h\phi_{kk^{-1}}(h^{-1}), 1_K) = (1_H, 1_K).$$

Likewise for fun we observe that $\phi_k^{-1} = \phi_{k^{-1}}$ so that

$$(x, k^{-1})(h, k) = (x(k^{-1}.h), k^{-1}k) = (\phi_k^{-1}(h^{-1})\phi_{k^{-1}}(h), 1_K) = (\phi_k(h^{-1}h), 1_K) = (1_H, 1_K)$$

as desired.

2. If we identify H with $\{(h, 1)\}$, then show that H is a *normal* subgroup of G. This helps explain the notation, the H is the normal factor in $H \rtimes K$.

Solution: It is easy to see that H is a subgroup and so the question is whether or not it is normal. We notice that

$$(h, k)(y, 1)(h, k)^{-1} = (h\phi_k(y), k)(\phi_k^{-1}(h^{-1}), k^{-1}) = (h\phi_k(y)\phi_k(\phi_k^{-1}(h^{-1})), kk^{-1}) = (h\phi_k(y)h^{-1}, 1_K)$$

which is still in H and so H is normal.

3. Let $H = \mathbb{Z}/4\mathbb{Z}$ and $K = \mathbb{Z}/2\mathbb{Z}$. Let the map $\phi : K \longrightarrow \mathrm{Aut}(Hs)$ be the map which sends [1] to the inversion bijection. In other words, $\phi([1])(k) = -k$ and $\phi([0])(k) = k$. Show that $H \rtimes K$ is isomorphic D_8 (the dihedral group on the square, which other books will denote by D_4).

Solution: We consider the element $r = ([1], [0])$ and $s = ([0], [1])$. Note we write $H \rtimes K$ multiplicatively even though H and K are additive. It is easy to check that $r^4 = ([0], [0])$ and $s^2 = ([0], [0])$. It is also easy to see that r and s span $H \rtimes K$. Next we verify that $rs = sr^{-1}$, note

$$rs = ([1], [0])([0], [1]) = ([1] + \phi_{[0]}([0]), [0] + [1]) = ([1], [1])$$

while

$$sr^{-1} = ([0], [1])([3], [0]) = ([0] + \phi_{[1]}([3]), [1] + [0]) = ([1], [1])$$

as desired.

This means that the surjective map from the free group on generators $\{r, s\}$ to $H \rtimes K$ has kernel which contains $\{r^r, s^2, rs - sr^{-1}\}$. But modding out by that kernel gives us D_8. Hence we have a surjective group homomorphism $D_8 \longrightarrow H \rtimes K$ between two groups each with 8 elements. It follows that this must be an isomorphism.

4. Suppose G is a group with K an subgroup of G and H a normal subgroup of G. Further suppose that $HK = G$ and that $H \cap K = \{1\}$. Let $\phi : K \longrightarrow \mathrm{Aut}(H)$ be the map which conjugates by k. In other words $\phi(k)(h) = khk^{-1}$. Show that $G \cong H \rtimes K$.

Hint: First show that every element of HK can be written uniquely as hk for some $h \in H$ and $k \in K$. Then define a map $HK \longrightarrow H \rtimes K$ sending $hk \mapsto (h, k)$. You need to show that this is a homomorphism (it is obviously a bijection).

Solution: Following the hint, suppose that $h'k' = hk$ are two different ways to write an element of HK. Thus $h^{-1}h' = kk'^{-1} \in H \cap K = \{1\}$ and so $h = h'$ and $k = k'$ and the first part of the hint is verified. We then get a bijective function $\phi : G = HK \longrightarrow H \rtimes K$. We will show it is a homomorphism. Choose $hk, h'k' \in G$. Since H is normal, $kH = Hk$ and so we can write $h'' = kh'k^{-1}$ for some $h'' \in H$. Then

$$\phi((hk)(h'k')) = \phi(hh''kk') = (hh'', kk') = (hkh'k^{-1}, kk') = (h\phi_k(h'), kk') = (h, k)(h', k') = \phi(hk)\phi(h'k')$$

which completes the proof.