WORKSHEET #2 – MATH 536
SPRING

DUE MONDAY, FEBRUARY 10TH

1. Show that there can be no simple group of order 108 with a subgroup of order 27. (We'll see shortly that there is always a subgroup of order 27).

Hint: Act on cosets.

Solution: Suppose G is a simple group with $|G| = 108$ and H is a subgroup with $|H| = 27$. Write $108 = 3^3 \cdot 2^2$. Let A be the set of left cosets of H. We have G acting on A by left multiplication. This gives us a map $G \longrightarrow S_4$, the target is size 24. The kernel of ϕ is non-trivial (by the sizes of the groups), and G acts transitively, so $\ker \phi \neq G$. Then $\ker \phi$ is a nontrivial normal subgroup.

2. Show that A_6 has no subgroup of order 72, you may use the fact that A_6 is simple.

Hint: Act on cosets.

Solution: If A_6 has such a subgroup H, consider the action of A_6 on the $(6!/2)/72 = (720/2)/72 = 5$ left cosets of H. Again we have a map $A_6 \longrightarrow S_5$. Note that the kernel is non-trivial since $|S_5| = 120$ and is not all of A_6 since A_6 acts transitively. This contradicts the simplicity of A_6.

3. Let S_i be a family of subgroups in some group G. Suppose that an element $x_i \in G$ is chosen for each i, and we form the intersection of all the cosets $x_i S_i$. Prove that this intersection is either empty or a coset of the intersection of all the S_i.

Solution: Let $S = \bigcap_i S_i$. Suppose that $x \in \bigcap_i x_i S_i$ and so the intersection is non-empty. In particular, $x \in x_i S_i$ for each i. Thus $xS_i = x_i S_i$ for each i. Hence $\bigcap_i x_i S_i = \bigcap_i x S_i = x \bigcap S_i = xS$ as desired.

4. Let H be a group with $|H| = 5^r 7^s$, $r, s \geq 0$. Suppose that H acts on a set A with 13 elements, prove that some element of A is fixed by this action.

Solution: Each orbit of $a \in A$ has size $|H : H_a|$. Each of those numbers are ≤ 13 and divide the size of H so the only possible orbit sizes are 5 and 7 and 1. It's easy to see that the only way this can happen is if one orbit has size 7, one has size 5 and one has size 1.

5. Consider $\sigma = (123)(4567) \in S_7$ and let $G = \langle \sigma \rangle$.
(a) Compute $|G|$.
(b) How many conjugates does σ have?
(c) What is the order of the centralizer of σ in S_7?
(d) How many distinct conjugates does G have in S_7?

Solution: (a) $|G| = |\sigma| = 3 \cdot 4 = 12$ since σ is a product of two disjoint cycles.

(b) The possible conjugates are a choice of a 3 cycle and a 4 cycle. Thus there are $\binom{7}{3}$ ways to break the numbers up. There are $2! = 2$ distinct 3 cycles on a 3-element set (ie, (123) or (132)). Likewise there are $3! = 6$ different 4-cycles on a 4 element set. Hence the answer is $\binom{7}{3} \cdot 2 \cdot 6 = 35 \cdot 2 \cdot 6 = 420$.

(c) We know the orbit size of σ is 420 and hence $420 = |G : C_G(\sigma)|$ and so $C_G(\sigma) = \frac{7!}{420} = 12$. Thus the centralizer has order 12.

(d) All the conjugates of G will also be cyclic and generated by elements of the same permutation type, so there are 420 possibilities, but some generate the same subgroup. We just have to figure out how many such generators already live in G (or any of its conjugates). Cyclic groups of order 12 have $\phi(12) = 4$ generators, and these are also of the same form in this case as can easily be verified. Thus we get $420/4 = 105$.