WORKSHEET #1 – MATH 536
SPRING

DUE MONDAY, FEBRUARY 3RD

Theorem 0.1 (Jordan-Hölder Theorem)**.** *If G is a finite group with $G \neq \{1\}$ the*

(1) *G has a composition series*
(2) *The composition factors in a composition series are unique up to ordering.*

First we prove the following.

1. Assume G has two composition series:

$$\{1\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_r = G \text{ and } \{1\} = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G$$

Prove that $r = 2$ and that the list of composition factors is the same.

*H*int: There are several steps (which to a certain extent can be done in various orders). For what i is it possible that $N_i \supseteq M_1$? Next intersect the first composition series with M_1, what can we conclude? Finally, multiply the first composition series by M_1 and then mod out by M_1. Use the second isomorphism theorem.

Solution: Now, it is easy to see that $H = M_1 \cap N_{r-1}$ is a proper normal subgroup of G (for normality, see **3.**). It is also a normal subgroup of the simple group M_1 so $H = M_1$ or $\{1\}$.

If $H = M_1$ then $N_{r-1} \supseteq M_1$ which implies that N_{r-1}/M_1 is a proper normal subgroup of the simple group G/M_1, hence $N_{r-1} = M_1$ and the statement is obvious.

If $H = \{1\}$ we clearly see that $M_1 N_{r-1}$ is a normal subgroup of G properly containing M_1 and so $M_1 N_{r-1} = G$ by the simplicity of G/M_1 again. Then $N_{r-1} \cong N_{r-1}/H \cong M_1 N_{r-1}/M_1 = G/M_1$ and so N_{r-1} is simple and hence $r = 2$. We also see that the composition factors of the $\{N_i\}$ composition series are simply the composition factors of the M_i composition series in reverse order.

2. Prove the existence of composition series. In fact, prove more generally that if $H \trianglelefteq G$, then there is a composition series of G where H is one of the terms.

Hint: This should be easy.

Solution: We may assume that $H \subsetneq G$ as the case where $H = G$ is equivalent to the case where $H = \{1\}$. We proceed by induction on $|G|$, the base case that $|G| = 2$ is obvious and then $|G|$ is already simple. Next, let N be a largest proper normal subgroup of G which contains H. Note that G/N is simple since if it isn't, and $M \trianglelefteq G/N$, $M \neq \{1\}, G/N$, then $\pi^{-1}(M)$ normal and is not equal to G or N. Since $|N| < |G|$ and $H \trianglelefteq N$, our inductive hypothesis completes the proof.

3. Prove an intersection of normal subgroups is normal.

Solution: Suppose that $\{N_\gamma\}_{\gamma \in \Gamma}$ is a collection of normal subgroups of G. Consider the diagonal map $G \longrightarrow \prod_{\gamma \in \Gamma}(G/N_\gamma)$ sending $g \mapsto \big(\ldots, g + N_\gamma, \ldots\big)$. The kernel of this map is easily seen to be $\bigcap_{\gamma \in \Gamma} N_\gamma$.

4. Now prove the uniqueness result of the Jordan-Hölder theorem. In particular, assume that

$$\{1\} = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_r = G \text{ and } \{1\} = M_0 \trianglelefteq M_1 \trianglelefteq \ldots \trianglelefteq M_s = G$$

are two composition series, prove that $r = s$ and that the composition factors are the same.

Hint: Proceed by induction on $\min\{r, s\}$. Apply the inductive hypothesis to $H = N_{r-1} \cap M_{s-1}$ (which we note is still normal). You can use a similar strategy to what was done in **2.**

Solution: We can assume $r, s \geq 2$ and also that the N_i form a composition series of minimal length (the base case is done) so that $r \leq s$. Note we can also assume that $N_{r-1} \not\subseteq M_{s-1}$ and $M_{s-1} \not\subseteq N_{r-1}$ otherwise we would be done by hypothesis. Hence $H \subsetneq N_{r-1}, M_{s-1}$. This also implies that $N_{r-1}M_{s-1} = G$ just as in **1**. Next observe that $N_{r-1}/H \cong M_{s-1}N_{r-1}/M_{s-1} = G/M_{s-1}$ is simple (as is $M_{s-1}/H \cong G/N_{r-1}$). Let $\{1\} \trianglelefteq N'_1 \trianglelefteq \ldots \trianglelefteq N'_{r-2} = H \trianglelefteq N_{r-1}$ be a composition series for N_{r-1}. Note that this must be of length $r - 1$ by induction. In particular, every composition series H has the same length of $r - 2$. But we can construct a similar composition series for H using

$$\{1\} \trianglelefteq M'_1 \trianglelefteq \ldots H = M_{s'-1} = M_{s'} = M_{s-1}.$$

but this also produces a composition series for H of length $s' - 1$. Hence $s' - 1 = r - 2$ and so M_{s-1} has a composition series of length $r - 1$ and thus $r = s$ as claimed. For the composition factors, using the inductive hypothesis to see that the composition factors of M_{s-1} are those of H concatenated with $M_{s-1}/H = G/N_{r-1}$. Hence the composition factors of $\{M_i\}$ are equal to

$$\Big(\text{composition factors of } H, G/N_{r-1}, G/M_{s-1}\Big)$$

and likewise the composition factors of $\{N_j\}$ are simply

$$\Big(\text{composition factors of } H, G/M_{s-1}, G/N_{r-1}\Big)$$

and hence the two sets coincide.