SOME RING THEORY PROBLEMS – MATH 536
SPRING

We will frequently use the following theorem.

Theorem 0.1. *Suppose R is a commutative ring with unity and I is an ideal. Then there is a bijection between the prime (respectively maximal) ideals of R/I and the prime (respectively maximal) ideals of R/I which contain I.*

This bijection is as follows where $\pi : R \longrightarrow R/I$ is the canonical surjective. If $P \subseteq R$ is prime contains *I, then $\pi(P) \subseteq R/I$ is also prime. Likewise if $Q \subseteq R/I$ is prime, then $\pi^{-1}(Q)$ is prime.*

1. Describe all the prime ideals of $\mathbb{Z}[x]$ which contain $\langle 3, x^2 + 3x + 5\rangle$.

Solution: Using the theorem above, these primes are in bijection with the primes of $\mathbb{Z}[x]/\langle 3, x^2 + 3x + 5\rangle \cong (\mathbb{Z}/3\mathbb{Z})[x]/\langle x^2 + 3x + 5\rangle$. Note that since we are working modulo 3, $x^2 + 3x + 5 = x^2 + 2$. Hence we are considering the ring $\mathbb{Z}/3\mathbb{Z}[x]/\langle x^2 + 2\rangle$. We see also that $x^2 + 2 = (x+1)(x+2)$. Hence any prime of $\mathbb{Z}/3\mathbb{Z}[x]/\langle x^2 + 2\rangle$ contains $0 = x^2 + 2$ and so must contain either $(x + 1)$ or $(x + 2)$.

Now, if Q is such a prime and $(x + 1) \in Q$ then $\langle x + 1\rangle \subseteq Q$. However, $\langle x + 1\rangle$ is maximal since if we quotient out by it we just get $\mathbb{Z}/3\mathbb{Z}$ (this is just setting $x = -1$). But then $Q = \langle x + 1\rangle$.

Likewise if Q is prime $(x+2) \in Q$ then $Q = \langle x+2\rangle$. Thus we just need to understand the inverse images of these two ideals in our original ring.

The inverse images are

$$\langle x + 1, 3, x^2 + 3x + 5\rangle \text{ and } \langle x + 1, 3, x^2 + 3x + 5\rangle.$$

and these are the two desired primes.

Alternate Solution: We see that the ideal $\langle 3, x^2+3x+5\rangle = \langle 3, x^2+3x+2\rangle = \langle 3,(x+2)(x+1)\rangle$ and so any prime Q containing it contains either $(x + 2)$ or $(x + 1)$. But note that $\langle 3, x + 2\rangle$ and $\langle 3, x + 1\rangle$ are maximal since their quotients are both $\mathbb{Z}/3\mathbb{Z}$ (mod out by 3 and then set $x = -2$ or $x = -1$). Thus either $Q = \langle 3, x + 2\rangle$ or $Q = \langle 3, x + 1\rangle$.

2. Let k be a field. Describe all prime ideals of $k[x, y]$ which contain $\langle x - 3, y^2 - 3y + xy - 1\rangle$.

Solution: As before we consider $k[x, y]/\langle x - 3, y^2 - 3y + xy - 1\rangle$. This time the first generator of the ideal $x - 3$ means that x is evaluated at 3. Hence this ring is isomorphic to

$$k[y]/\langle y^2 - 3y + 3y - 1\rangle = k[y]/\langle y^2 - 1\rangle.$$

Now $y^2 - 1 = (y - 1)(y + 2)$ and so any prime Q of $k[y]/\langle y^2 - 1\rangle$ contains either $y - 1$ or $y + 1$. As before, since $\langle y - 1\rangle \subseteq k[y]/\langle y^2 - 1\rangle$ is maximal (with quotient field k) and likewise since $\langle y + 1\rangle$ is maximal, we see that a prime of this quotient ring is equal to either $\langle y + 1\rangle$ or $\langle y - 1\rangle$. Hence using the theorem we started with, the primes containing $\langle x - 3, y^2 - 3y + xy - 1\rangle$ are exactly:

$$\langle y - 1, x - 3, y^2 - 3y + xy - 1\rangle \text{ and } \langle y + 1, x - 3, y^2 - 3y + xy - 1\rangle.$$

Note that this argument is really the same as the one over $\mathbb{Z}[x]$.

3. Find a ring R with exactly 3 prime ideals.

Solution: We use the theorem above. The ring $\mathbb{Z}/30\mathbb{Z}$ works. Note that the primes of $\mathbb{Z}/30\mathbb{Z}$ are in bijection with the primes of $\mathbb{Z}$ which contain 30. These are just $\langle 2\rangle, \langle 3\rangle$ and $\langle 5\rangle$.

Alternate Solution: Consider k a field not of characteristic 2 and form $k[x]/\langle x(x-1)(x-2)\rangle$. Using the same logic as in the first solution, the prime of this ring are just $\langle x\rangle, \langle x-1\rangle$ and $\langle x-2\rangle$ (you can see that each of these ideals is prime since their quotients are all isomorphic to k which is an integral domain).

4. Suppose that A and B are commutative rings with unity. Show that the primes of $A \oplus B$ are in bijections with (primes of A) $\coprod$(primes of B)

Solution: We have surjective ring homomorphisms $p_1 : A \oplus B \longrightarrow A$ and $p_2 : A \oplus B \longrightarrow B$.

Suppose first that Q is a prime ideal in $A \oplus B$. Then $(1, 0) \cdot (0, 1) = (0, 0) \in Q$. So $(1, 0) \in Q$ or $(0, 1) \in Q$. In the first case, $(A, 0) \subseteq Q$ and since $\pi_2(Q) \subseteq B$ is a prime ideal, we see that $(A, \pi_2(Q)) \subseteq Q$. But in fact this must be equality since this is the largest ideal of $A \oplus B$ whose second term is $\pi_2(Q)$. Hence Q corresponds to $\pi_2(Q)$.

By symmetry, if $(0, 1) \in Q$ we see that $Q = (\pi_1(Q), B)$ and thus that Q corresponds to $\pi_1(Q)$. We have thus obtained a prime of (primes of A) $\coprod$(primes of B) from a prime of $A \oplus B$.

Conversely, if Q_1 is a prime of A, then consider $Q = (Q_1, B) = Q_1 \oplus B$. This is obviously an ideal of $A \oplus B$ and if we form the quotient $A \oplus B/Q_1 \oplus B$ we obtain $A/Q_1 \oplus 0 \cong A/Q_1$ which is a domain. Hence Q is a prime of $A \oplus B$. Likewise if Q_2 is a prime of B then $Q' = A \oplus Q_2$ is a prime of $A \oplus B$. This identification is clearly inverse to the identification above and hence we have obtained a bijection as desired.

5. Fix a prime integer p, given the fact that the ring of p-adic integers $\mathbb{Z}_p$ have exactly two prime ideals $\langle 0\rangle$ and $\langle p\rangle$, find a ring with exactly 2 maximal ideals and 3 prime ideals.

Solution: Consider $\mathbb{Z}_p \oplus \mathbb{C}$. Use problem **4.** to see that there are exactly 3 prime ideals $\langle 0\rangle \oplus \mathbb{C}$, $\langle p\rangle \oplus \mathbb{C}$ and $\mathbb{Z}_p \oplus \langle 0\rangle$. Since these are all of the primes, they are maximal if and only if they are maximal elements of this set. Thus the second two primes are maximal.

We really only used the fact that $\mathbb{Z}_p$ had exactly two prime ideals, and only one is maximal. There are other ways to cook up such rings. For instance if $R = \mathbb{C}[x]$ and $W = \{f \in R \mid f(1) \neq 0\}$ then $W^{-1}R$ has exactly two primes, $\langle 0\rangle$ and $\langle x\rangle$.

6. Consider the ring $R = \mathbb{Z}[x]$. Find a prime ideal $Q \subseteq R$ such that R/Q has 4 elements.

Solution: We need a field with $4 = 2^2$ elements, so it's natural that $2 \in Q$. Note that $\mathbb{Z}[x]/\langle 2\rangle = \mathbb{Z}/2\mathbb{Z}[x]$ and so we should include an irreducible polynomial of degree 2 (since it is then easy to see that there are exactly 4 cosets – each represented by a unique polynomial of degree < 2). In characteristic 2, the only such polynomial is $x^2 + x + 1$. Hence we can make $Q = \langle 2, x^2 + x + 1\rangle$.

What if we wanted R/Q to have 9 elements?

7. Describe the maximal ideals $\mathfrak{m}$ in $\mathbb{Z}[x]$ that contain 30 and x^2+1. Give explicitly two generators for each such maximal ideal.

Solution: Note that any such maximal ideal $\mathfrak{m}$ contains 2, 3 or 5. We handle each case separately.

If $2 \in \mathfrak{m}$, then using the theorem 0.1 we can find the maximal ideals $\mathfrak{m}'$ of $\mathbb{Z}/2\mathbb{Z}[x]$ which contain x^2+1. Since $x^2+1=(x+1)^2$ in this ring, any such maximal ideal $\mathfrak{m}'$ also contains $x+1$. Hence $\mathfrak{m}$ also contains $x+1$. But $\langle 2, x+1\rangle$ is already maximal.

If $3 \in \mathfrak{m}$ then again working in $\mathbb{Z}/3\mathbb{Z}[x]$ we see that x^2+1 is irreducible so that $\langle x^2+1\rangle$ is already maximal. Thus $\mathfrak{m}=\langle 3, x^2+1\rangle$.

Finally if $5 \in \mathfrak{m}$ we work in $\mathbb{Z}/5\mathbb{Z}[x]$ and there $x^2+1=(x+2)(x+3)$. If $\mathfrak{m}'$ is a maximal ideal in $\mathbb{Z}/5\mathbb{Z}[x]$ containing x^2+2, we see that $x+2$ or $x+3$ is in $\mathfrak{m}'$. But $\langle x+2\rangle$ and $\langle x+3\rangle$ are already maximal in $\mathbb{Z}/5\mathbb{Z}[x]$. Thus $\mathfrak{m}=\langle 5, x+2\rangle$ or $\mathfrak{m}=\langle 5, x+3\rangle$.

Combining this information together, we see that

$$\langle 2, x+1\rangle, \langle 3, x^2+1\rangle, \langle 5, x+2\rangle, \langle 5, x+3\rangle$$

are all the maximal ideals of $\mathbb{Z}[x]$ which contain 30 and x^2+1.