SOLUTION TO A HOMEWORK PROBLEM

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Problem: Suppose that $\{x_i\}$ are distinct representatives of the conjugacy classes of a finite group G. Further suppose that $x_i x_j = x_j x_i$ for all i, j. Then show that G is Abelian.

Solution: We let G act on itself by conjugation. For each x_j we note that $\mathrm{Stab}_G(x_j) \supseteq \{x_i\}$. For any $y \in G$, consider $y\mathrm{Stab}_G(x_j)y^{-1}$. We notice that these elements stabilizes yx_jy^{-1}, indeed if $yzy^{-1} \in y\mathrm{Stab}_G(x_j)y^{-1}$ then $yzy^{-1}yx_jy^{-1}yz^{-1}y^{-1} = yzx_jz^{-1}y^{-1} = yx_jy^{-1}$. On the other hand, given any $g \in G$, notice that $g = yx_iy^{-1}$ for some $y \in G$ and one of the given x_i. Hence $g \in y\mathrm{Stab}_G(x_j)y^{-1}$. It follows that every element of G is contained in a conjugate of $H = \mathrm{Stab}_G(x_j)$.

We will show that this forces $H = G$ which will imply that $x_j \in Z(G)$. This will prove that the conjugacy class of x_j is a single element, and since this holds for each x_j, it holds for each conjugacy class of G, which will prove that G is Abelian.

Ok, so suppose that $H \neq G$. Then G acts on the set of conjugates of H by conjugation. Note that under this action, $\mathrm{Stab}_G(H) = N_G(H)$. In particular, it follows that the number of conjugates of H (ie the size of the orbit of H) is $|G : N_G(H)|$. Now we count the number of elements in each conjugate of H. Of course, $G = \bigcup_{g \in G} gHg^{-1}$ and so

$$|G| \leq |G : N_G(H)| \cdot |H|.$$

Note also that H cannot be normal so $|G : N_G(H)| > 1$.

In particular, the number of conjugates times the size of any conjugate. But now observe that we have overcounted, each conjugate contains the identity e and hence

$$|G| \leq |G : N_G(H)| \cdot |H| - |G : N_G(H)| + 1.$$

But now observe that

$$|G : N_G(H)| \cdot |H| - |G : N_G(H)| + 1 \leq |G : N_G(H)| \cdot |N_G(H)| - |G : N_G(H)| + 1 = |G| - |G : N_G(H)| + 1 < |G|$$

a contradiction. Hence $H = G$ and we are done.