ALTERNATE PROOF OF A THEOREM FROM THE BOOK – MATH 536 SPRING

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Theorem 0.1. *Suppose R is a PID and $M = R^{\oplus m}$. Suppose that $N \subseteq M$ is an R-submodule. Then $N \cong R^{\oplus n}$ for some $n \leq m$.*

Proof. We follow the book to start. Set $\Sigma = \{\phi(N) \subseteq R \mid \phi \in \mathrm{Hom}_R(M, R)\}$ and let $I_\phi \subseteq R$ be a maximal element of Σ with $I = \phi(N)$ for some fixed ϕ. We know $I_\phi = \langle a_\phi \rangle$ where $a_\phi = \phi(y)$ for some $y \in N \subseteq M$. Obviously if $N \neq 0$ then $\phi(N) \neq 0$ for some $\phi \in \mathrm{Hom}_R(M, R)$ (consider the projections) and so since I_ϕ is maximal we may assume that $I_\phi \neq 0$.

Claim 0.2. *$\psi(y) \in I_\phi$ for every $\psi \in \mathrm{Hom}_R(M, R)$.*

Proof of claim. Consider the ideal $\langle \psi(y), \phi(y) \rangle = \langle \psi(y) \rangle + I_\phi = J_\psi = \langle b_\psi \rangle$. Obviously $b_\psi = r\psi(y) + s\phi(y)$ for some $r, s \in R$ and so $J_\psi = \langle (r\phi + s\psi)(y) \rangle \subseteq (r\phi + s\psi)(N)$. But J_ψ contains I_ϕ and so $J_\psi = I_\phi$ by maximality. Hence $\psi(y) \in I_\phi$ as claimed. □

We return to the main proof. Our next goal is to construct an element $y' \in M$ such that y is a multiple of y' and such that $\phi(y') = 1 \in R$. Note that y' will not be in N.

We let $\pi_i : M = R^{\oplus m} \longrightarrow R$ to be the projection onto the ith component. Then $\pi_i(y) \in I_\phi = \langle a_\phi \rangle$ for each i. Hence we can write $\pi_i(y) = b_i a_\phi$ for each i. Write $y' = b_1 e_1 + \ldots + b_m e_m$ where the e_i are the canonical basis elements in $M = R^{\oplus m}$. Now notice that

$$a_\phi y' = a_\phi b_1 e_1 + \ldots + a_\phi b_m e_m = \pi_1(y) e_1 + \ldots + \pi_m(y) e_m = y.$$

Also $a_\phi = \phi(y) = \phi(a_\phi y') = a_\phi \phi(y')$ and so since R is an integral domain we see that $\phi(y') = 1$ as claimed.

Lemma 0.3. *If we have a map $\phi : M \longrightarrow R$ and an element $y' \in M$ such that $\phi(y') = 1$ then $M \cong y'R \oplus K$ where $K = \ker \phi$.*

Proof of lemma. Consider the map $\rho : y'R \oplus K \longrightarrow M$ which sends $(y'r, k) \mapsto y'r + k$. If $(y'r, k) \in \ker \rho$ then $y'r + k = 0$ and so $y'r \in K$. But then $\phi(y'r) = r = 0$. Hence $(y'r, k) = (0, 0)$ and ρ injects. On the other hand given any $z \in M$ note that $\phi(\phi(z)y') = \phi(z)\phi(y') = \phi(z)$. Thus $z - \phi(z)y' \in \ker \phi$. But now write $z = \phi(z)y' + (z - \phi(z)y') \in \mathrm{Image}(\rho)$. This shows that ρ surjects as claimed. □

Using the lemma and noting that $\langle a_\phi \rangle = a_\phi R \cong R$ as R-modules. We also see that $\phi|_N$ shows us that

$$N \cong \langle y \rangle \oplus \ker(\phi|_N) = \langle y \rangle \oplus (K \cap N).$$

From here on now we can proceed by induction on the rank of N. The base case when N is rank 0 is trivial since then $N = 0$ (since N is torsion free).

We construct ϕ as above and write $M = \langle y' \rangle \oplus K$ and $N = \langle a_\phi y' \rangle \oplus (K \cap N)$. Obviously $K \cap N$ has rank lower than N (since $\langle a_\phi y' \rangle$ has rank 1). Thus by induction $K \cap N$ is free. But then N is a direct sum of free modules and so it is also free. □

Remark 0.4. As the book points out, by carefully stepping through the induction, one actually observes that one can choose a basis $y'_1, \ldots, y'_m$ for M such that $a_1 y'_1, a_2 y'_2, \ldots, a_n y'_n$ is a basis for N (for some $a_i \in R$).