WORKSHEET #1 (IDENTIFYING GROUPS) – MATH 435

Below are sets with potential binary operations (ie, a way to combine elements). Determine if each set is (or is not) a group and prove your answer. If it is a group, is it Abelian?

1. For a fixed integer n, the numbers $\{0, 1, 2, \ldots, n-1\}$. The binary operation is multiplication modulo n.

Solution: This is not a group. While 1 is indeed a multiplicative identity, the number 0 does *not* have a multiplicative inverse since the equation $x \cdot 0 = 1$ doesn't have a solution.

2. The set of real-valued $n \times n$ matrices with positive determinant. The binary operation is matrix multiplication.

Solution: From the (un)assigned homework, the determinant of a product of matrices is the product of the determinants. Thus $\det(A \cdot B) = \det(A) \cdot \det(B)$. In particular, if $\det(A), \det(B) > 0$, then so is $\det(A \cdot B)$ and so matrix multiplication is a binary operation on this set (in other words, the closure property is satisfied). Matrix multiplication is always associative. The identity matrix has determinant 1 and so it is in the set and so the set has identity. Finally, we notice that if $\det(A) > 0$, then $\det(A) \neq 0$ and so A is invertible and also that $\det(A^{-1}) = \frac{1}{\det(A)} > 0$ which implies that inverses are in our set. Thus this set is a group under matrix multiplication.

3. The set of real-valued 2×2 matrices with integer determinant. The binary operation is matrix multiplication.

Solution: This is not a group although it does have identity $\mathrm{Id}_{2\times 2}$. It however does not have inverses. If $\det(A) = 4$ (for example, if $A = 2 \cdot \mathrm{Id}_{2\times 2}$), then the only possible inverse is the ordinary inverse A^{-1} (in our example, $1/2 \cdot \mathrm{Id}_{2\times 2}$) since inverses in matrix multiplication are unique. But $\det(A^{-1}) = \frac{1}{4} \notin \mathbb{Z}$ and so A^{-1} is not in our set.

4. Fix a group G and consider the set $H = \{g \in G \,|\, ga = ag \ \forall a \in G\}$. The binary operation on H is the binary operation from the group G.

Solution: We will show that H is indeed a group. First we show that the operation on H is really a binary operation (ie, that it is closed under this operation). Take $h, h' \in H$ and fix any $a \in G$. Then $(hh')a = h(h'a) = h(ah') = (ha)h' = (ah)h' = a(hh')$ by using (respectively) the associativity of G, the defining property of H, the associativity of G again, the defining property of H again, and finally the associativity of G one last time.

We note that multiplication in H itself is associative because the multiplication in G is. We notice that for any $a \in G$, $ea = a = ae$ and so the identity element of G is in H and thus H itself has an identity. Before proving inverses, we need a Lemma **Lemma.** *If $a, b \in A$ are elements of a group A, then $(ab)^{-1} = b^{-1}a^{-1}$.*

Proof. Note simply that

$$(ab)(b^{-1}a^{-1} = a((bb^{-1})a^{-1}) = a(ea^{-1}) = aa^{-1} = e$$

and

$$(b^{-1}a^{-1})(ab) = b^{-1}((a^{-1}a)b) = b^{-1}(eb) = b^{-1}b = e.$$

This ends the proof of the Lemma. □

Finally if $g \in H$, and $a \in G$, then notice that

$$g^{-1}a = g^{-1}(a^{-1})^{-1} = (a^{-1}g)^{-1} = (ga^{-1})^{-1} = (a^{-1})^{-1}g^{-1} = ag^{-1}$$

where we use the lemma and the fact that $g \in H$.

5. The numbers $\{1, 2, \ldots, 7\}$. The (potential) binary operation is mutliplication modulo 8.

Solution: This is not a group since $4 \cdot 2 = 0$ modulo 8 which is not in the set.