SOME SOLUTIONS TO HOMEWORK #5

MATH 435 – SPRING 2012

Certainly there are many correct ways to do each problem.

#2 on page 87. *Let G be the group of all real-valued functions on the unit interval $[0,1]$, where we define, for $f, g \in G$, addition by $(f+g)(x) = f(x) + g(x)$ for every $x \in [0,1]$. If $N = \{f \in G | f(\frac{1}{4}) = 0\}$, prove that $G/N \simeq$ real numbers under $+$.*

Proof. Set $\mathbb{R}$ to be the real numbers under addition. Consider the map $\Phi : G \to \mathbb{R}$ defined by the rule $\Phi(f) = f(\frac{1}{4})$. Note that $\Phi(f+g) = (f+g)(\frac{1}{4}) = f(\frac{1}{4}) + g(\frac{1}{4}) = \Phi(f) + \Phi(g)$ and so Φ is a homomorphism. Now choose any number $y \in \mathbb{R}$ and consider the constant function $f(x) = y$. Then $\Phi(f) = f(\frac{1}{4}) = y$ and so Φ is surjective.

Therefore, by the first homomorphism theorem $G/\ker\Phi \simeq \mathbb{R}$. On the other hand, clearly $\ker\Phi = \{f \in G \mid \Phi(f) = 0\}$ is equal to N (since $\Phi(f) = f(\frac{1}{4})$ and so the problem is complete. $\square$

#4 on page 87. *If G_1, G_2 are two groups and $G = G_1 \times G_2$ where we define multiplication component wise, show that*

(a) *$N = \{(a, e_2) | a \in G_1\}$ is a normal subgroup of G.*
(b) *$N \simeq G_1$.*
(c) *$G/N \simeq G_2$.*

Proof. Consider the map $\Phi : G \to G_2$ defined by the rule, $\Phi\big((a,b)\big) = b$. Note that $\Phi\big((a,b)(c,d)\big) = \Phi\big((ac,bd)\big) = bd = \Phi\big((a,b)\big)\Phi\big((c,d)\big)$. Therefore the map is a homomorphism. Note for any $b \in G_2$, $\Phi\big((e_1,b)\big) = b$ and so Φ is surjective.

Clearly, $N = \ker\Phi$ so that N is a kernel and hence normal. This prove (a). On the other hand, notice that we have a map $\psi : N \to G_1$ defined by $\psi\big((a,e_2)\big) = a$. As above, this is a homomorphism, and it is easy to see that this one is in fact bijective (I leave that to you). Thus (b) is also done.

Finally, we handle (c), note that the first homomorphism theorem and (a) easily completes the proof since $G_2 \simeq G/\ker\Phi = G/N$. $\square$

#7 on page 88. *If ϕ is a homomorphism of G onto G' and N is a normal subgroup of G, show that $\phi(N)$ is also normal.*

Proof. We already know $\phi(N)$ is a subgroup, so we have to prove that it is normal. Choose $y = \phi(x) \in \phi(N)$ (so $x \in N$) and choose $g' \in G'$. Since ϕ is surjective, there exists $g \in G$ such that $\phi(g) = g'$. Note that $gxg^{-1} \in N$ since N is already normal. Then

$$g'yg'^{-1} = \phi(g)\phi(x)\phi(g)^{-1} = \phi(gxg^{-1}) \in \phi(N)$$

which completes the proof. $\square$

#1 on page 117. *Show that if σ, τ are two disjoint cycles, the $\sigma\tau = \tau\sigma$.*

Proof. Lets suppose these are cycles in some S_n. Choose $x \in \{1, \dots, n\}$. Then there are three possibilities.

(1) x appears in σ but not τ. In this case $\sigma(x)$ also does not appear in τ and so

$$\sigma\tau(x) = \sigma(x) = \tau\sigma(x)$$

(2) x appears in τ but not σ. This case is symmetric to (1) and so the proof will be left to the reader that $\sigma\tau(x) = \tau\sigma(x)$.

(3) x appears in neither σ nor τ. In this case,

$$\sigma\tau(x) = \sigma(x) = x = \tau(x) = \tau\sigma(x)$$

In any case, $\sigma\tau = \tau\sigma$ completing the proof. □

#3 on page 117. *Express as the product of disjoint cycles and find the order.*

Proof. (a) $(12357)(2476) = (124)(3576)$ which has order $3 \cdot 4 = 12$
(b) $(12)(13)(14) = (1432)$ which has order 4.
(c) $(12345)(12346)(12347) = (1473625)$ which has order 7.
(d) $(123)(132) = e = ()$ which has order 1 as it is the identity.
(e) $(123)(3579)(123)^{-1} = (123)(3579)(321) = (1579)$ which has order 4.
(f) $(12345)^3 = (12345)(12345)(12345) = (14253)$ which has order 5.

□

#6 on page 117. *Find a shuffle of a deck of 13 cards that requires 42 repeats to return the cards to the original order.*

Proof. Consider the shuffle which is made up of 2 cycles, the first reorders the first 7 cards (moving each card one over), the second (disjoint) cycle does the same to the final 6 cards. □