MATH 435, FINAL EXAM

Your Name

- You have 2 hours to do this exam.
- No calculators!
- No notes!
- For proofs/justifications, please use complete sentences and make sure to explain any steps which are questionable.
- All rings will be assumed to be commutative, associative and with unity.
- Good luck!

Problem	Total Points	Score
1	20	
2	20	
3	18	
4	12	
5	15	
6	15	
EC	10	
Total	100	

1. Definitions and short answers I.

(a) Prove that a group homomorphism $\phi : A \to B$ is injective if $\ker \phi = \{e_A\}$. (4 points)

(b) Consider $\mathbb{Z}_{\mathrm{mod} 2}$ acting on $\mathbb{Z}$ by the rule $0.x = x$ and $1.x = -x$. Find the orbit of 7. (4 points)

(c) Consider S_3 acting on itself by conjugation, compute the stabilizer of (123). (4 points)

(d) Given two ideals $I, J \subseteq R$, where R is a commutative associative ring with unity, prove that $I \cap J$ is also an ideal of R. (4 points)

(e) Explain why every subgroup of an Abelian group is normal. (4 points)

2. Definitions and short answers II.

(a) Define the term *splitting field.* (4 points)

(b) Suppose that R is an integral domain and $x \in R$ is a prime element. Prove that x is irreducible. (4 points)

(c) Give an example of an integral domain of characteristic p, for p a prime, which is not a field. (4 points)

(d) Suppose that $F \subseteq E$ is an extension of fields. Write down a precise definition of $[E : F]$ and give an example where that number is 8. (4 points)

(e) Using straight-edge, compass and a given unit length, is it possible to construct a length which is a root of the equation $x^5 + 9x^2 + 6x + 3$? Justify your answer. (4 points)

3. Consider the finite field $\mathbb{Z}/\langle 2\rangle = \mathbb{F}_2$. Here $\langle\cdot\rangle$ just means the ideal generated by "$\cdot$".
(a) Explicitly construct $\mathbb{F}_4$ and write down complete addition and multiplication tables for $\mathbb{F}_4$. (8 points)

(b) Find an irreducible polynomial $p(x) \in \mathbb{F}_4[x]$. What field is $\mathbb{F}_4[x]/\langle p(x)\rangle$ isomorphic to? What is the degree $[\mathbb{F}_4[x]/\langle p(x)\rangle : \mathbb{F}_2]$? (6 points)

(c) Find a polynomial $g(x) \in \mathbb{F}_2[x]$ that is irreducible in $\mathbb{F}_2[x]$ but splits in $\mathbb{F}_4[x]$. Factor it explicitly. (4 points)

4. Suppose that K is an algebraic extension of F (meaning every element of K is algebraic over F), $F \subseteq K$ and that L is an algebraic extension of K. Prove that $F \subseteq L$ is also an algebraic extension. (12 points)

Hint: Not every algebraic extension is finite (otherwise this problem would be very easy) although we do know, and you can use without proof, that every finite extension is algebraic. Choose $t \in L$, since L is algebraic over K, t must be a root of some polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ with coefficients in K. Consider $F(a_0, a_1, \ldots, a_n)$ and show it is a finite extension of F. Now complete the proof...

5. Prove that every finite group of order n is isomorphic to a subgroup of some group S_n. (15 points)

6. Suppose that k is a field and prove that $k[x]$ is a PID. Make sure to also explain why it is an integral domain. (15 points)

(EC) Suppose $F \subseteq E$ is a field extension in characteristic zero. Consider the group $G = \mathrm{Aut}(E/F)$ and suppose that $H \subseteq G$ is a subgroup. Show directly that the set

$$E_H = \{a \in E | \sigma(a) = a, \forall \sigma \in H\}$$

is a field (3 points). Further prove that $\mathrm{Aut}(E/E_H)$ is a subgroup of G which contains H. (2 points)

Recall some facts about fields extensions.

(i) For any any finite degree field extension $k \subseteq K$ of characteristic zero, we have $|\mathrm{Aut}(K/k)| \leq [K : k]$.

(ii) If $k \subseteq K$ is the splitting field of some polynomial $g(x) \in k[x]$ where k is of characteristic zero, then $k \subseteq K$ is a Galois extension, meaning that $|\mathrm{Aut}(K/k)| = [K : k]$.

Now for the problem. Suppose that $F \subseteq E$ is the splitting field of some polynomial $f(x) \in F[x]$. Prove directly that $\mathrm{Aut}(E/E_H) = H$. (5 points)