EXTRA CREDIT OVER SPRING BREAK #2

DUE FRIDAY MARCH 16TH

Throughout this assignment, we will *ALWAYS* be dealing with Abelian groups written under addition.

1. Finitely generated Abelian groups

Definition 1.1. Suppose that A is an Abelian group under addition. We say that A is *finitely generated* if there exist finitely many elements $a_1, \dots, a_k \in A$ such that every element $a \in A$ can be written as:

$$a = n_1 a_1 + \dots + n_k a_k$$

for some integers $n_i \in \mathbb{Z}$.

A finitely generated Abelian group should be viewed as something like a vector space, with the finite generating set being something like a *spanning set*. Linear dependence is much more subtle however, and it is not true that a minimal generating set is linearly independent.

Exercise 1.2. Prove that every cyclic group (under addition) is a finite generated Abelian group. (1 point)

Exercise 1.3. Suppose that A_1 and A_2 are finitely generated Abelian groups. Prove that $A_1 \times A_2 = \{(a_1, a_2) \,|\, a_1 \in A_1, a_2 \in A_2\}$ is also a finitely generated Abelian group where the group operation is performed componentwise.[1] (1 point)

Exercise 1.4. Consider $\mathbb{Q}$ under addition. Prove that $\mathbb{Q}$ is *NOT* a finitely generated Abelian group. (1 point)

Consider now the following theorem:

Theorem 1.5. *Suppose that G is a finitely generated Abelian group generated by $a_1, \dots, a_k$. Then there exists a surjective group homomorphism:*

$$\Phi : \mathbb{Z}^k = \mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z} \to G$$

which sends $(n_1, \dots, n_k)$ to $n_1 a_1 + \dots + n_k a_k$.

Proof. The map is clearly surjective, by the definition of a finitely generated Abelian group. We need to prove it is also group homomorphism. But simply note that

$$\begin{array}{rl} & \Phi\Big((n_1, \dots, n_k) + (m_1, \dots, m_k)\Big) \\ = & \Phi\Big((n_1 + m_1, \dots, n_k + m_k)\Big) \\ = & (n_1 + m_1)a_1 + \dots + (n_k + m_k)a_k \\ = & \big(n_1 a_1 + \dots + n_k a_k\big) + \big(m_1 a_1 + \dots m_k a_k\big) \\ = & \Phi\Big((n_1, \dots, n_k)\Big) + \Phi\Big((m_1, \dots, m_k)\Big). \end{array}$$

which completes the proof. ∎

We will also need the following fact which I will not prove.

[1] $(a_1, a_2) + (a_1', a_2') = (a_1 + a_1', a_2 + a_2')$

Theorem 1.6. *Every subgroup N of a finitely generated Abelian group A is also a finitely generated Abelian group.*

Proof. I will skip this, you may take it on faith or look it up. $\square$

Now we come to a subtle point. Suppose that A is a finitely generated Abelian group and $\Phi : \mathbb{Z}^k \to A$ is as above. Then $\ker \Phi$ is also a finitely generated Abelian group, and so we have another surjective map $\Psi : \mathbb{Z}^l \to \ker(\Phi)$. By composition, we have a map Γ,

$$\mathbb{Z}^l \xrightarrow{\Psi} \ker(\Phi) \hookrightarrow \mathbb{Z}^k$$

from $\mathbb{Z}^l$ to $\mathbb{Z}^k$.

Exercise 1.7. View the elements of $\mathbb{Z}^l$ as column vectors and explain why this map Γ can be identified with a $k \times l$ matrix with integer entries (1 point)

Exercise 1.8. Prove that $\mathbb{Z}^k/\Gamma(\mathbb{Z}^l)$ is isomorphic to A.

We call this matrix M.

If we do one of the following 6 operations on M, we get a new $k \times l$ matrix:

Column replacement: Add an integer multiple of one column to another column.
Column interchange: Switch two columns.
Column scaling: Scale a column by -1.
Row replacement: Add an integer multiple of one row to another row.
Row interchange: Switch two rows.
Row scaling: Scale a row by -1.

After doing one (or more) of these operations, each of the resulting matrices M' also gives us a map $\Gamma' : \mathbb{Z}^l \to \mathbb{Z}^k$. Then consider the quotient group:

$$\mathbb{Z}^k/\Gamma'(\mathbb{Z}^l).$$

Exercise 1.9. Show that if you do any single operation above, then

$$\mathbb{Z}^k/\Gamma'(\mathbb{Z}^l) \simeq A.$$

This is quite involved, although once you get the hang of it, it is not too hard. (3 points)

Suppose that after doing some series of column and row operations as above, you obtain a matrix N with corresponding map Γ_N. We notice that $\mathbb{Z}^k/\Gamma_N(\mathbb{Z}^l)$ is still going to be isomorphic to A.

Exercise 1.10. Suppose further that N is a diagonal matrix (or at least a matrix which is zero in every entry a_{ij} for $i \neq j$). Prove that A is isomorphic to $(\mathbb{Z}/f_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/f_k\mathbb{Z})$ for some integers f_i, some of which are possibly zero. (2 points)

Finally, we come to the big theorem.

Theorem 1.11. *Suppose that A is a finitely generated Abelian group. Then A is isomorphic to $(\mathbb{Z}/f_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/f_k\mathbb{Z})$ for some integers f_i (possibly zero).*

Exercise 1.12. Prove the theorem by explaining how to use the column and row operations described above to transform any matrix into a diagonal (enough) matrix. (2 points).