MATH 435, EXAM #1

Your Name

- You have 50 minutes to do this exam.
- No calculators!
- No notes!
- For proofs/justifications, please use complete sentences and make sure to explain any steps which are questionable.
- Good luck!

Problem	Total Points	Score
1	30	
2	25	
3	25	
4	20	
EC	10	
Total	100	

1. Definitions and short answers.
(a) State Lagrange's theorem. (5 points)

Solution: If G is a finite group and H is a subgroup of G, then the order of H divides the order of G.

(b) Give an example of a function $f : G \to H$ between two groups that is *NOT* a homomorphism. (5 points)

Solution: Consider $G = H = \mathbb{Z}$ under addition. Then consider the function $f(x) = x^2$. We observe that $f(x+y) = (x+y)^2 \neq x^2+y^2 = f(x)+f(y)$, at least in general, so that this function is not a homomorphism.

(c) Give an example of an element of order 3 in a group of order 6. (5 points)

Solution: The element 2 has order three inside $\mathbb{Z}_{\mathrm{mod} 6}$.

(d) In class, we showed that every group G is isomorphic to a subgroup of $A(S)$ for some set S. Define the term $A(S)$ and state what set S was used in the proof of this fact. (5 points)

Solution: $A(S)$ denotes the set of *bijective* functions from S to S. In that problem I mentioned, we set $S = G$.

(e) Assume that the kernel of a group homomorphism is a subgroup, prove that the kernel of a group homomorphism is a *normal* subgroup. (5 points)

Solution: Suppose that $\phi : G \to G'$ is a group homomorphism and that K is the kernel. We need to show that $xKx^{-1} \subseteq K$ for all $x \in G$. Fix an $x \in G$ and $k \in K$, we need to show that $xkx^{-1} \in K$. In other words, we need to show that $\phi(xkx^{-1}) = e_{G'}$. But

$$\phi(xkx^{-1}) = \phi(x)\phi(k)\phi(x^{-1}) = \phi(x)e_{G'}\phi(x^{-1}) = \phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_{G'}$$

as desired. This completes the proof.

(f) Prove that every group of order 5 is cyclic using Lagrange's theorem. (5 points)

Solution: Suppose that G is a group of order 5. Since G has more than one element, we choose $x \in G$, $x \neq e$. Consider now $H = \langle x \rangle \subseteq G$. This is a subgroup of G. By Lagrange's theorem, the order of H is either 1 or 5. Now, H contains x and since it is a subgroup, it also contains e. Thus $|H| \geq 2$ and so $|H| = 5$. But then $G = H$ and since H is cyclic, so is G.

2. Consider G the group of rotations of a hexagon, $G = \{e, r60, r120, r180, r240, r300\}$.
(a) Show that G is cyclic and identify all the generators. (6 points)

Solution: Note that $e = r60^6$, $r60 = r60^1$, $r120 = r60^2$, $r180 = r60^3$, $r240 = r60^4$ and $r300 = r60^5$. Thus $G = \langle r60 \rangle$ is cyclic. It is easy to see that the only generators of G are $r60$ and $r300 = r60^{-1}$. All the other elements are $r60$ raised to a power not coprime to 6.

(b) Write down the elements of $H = \langle r180 \rangle$, the cyclic subgroup generated by rotation by 180 degrees. Also write down all of the distinct cosets of H. (7 points)

Solution: Note

- $H = \langle r180 \rangle = \{e, r180\} = eH = r180H$.
- $r60H = \{r60, r240\} = r240H$.
- $r120H = \{r120, r300\} = r300H$.

And so there are 3 distinct cosets.

(c) Write down a complete multiplication table for G/H. (12 points)

Solution:

	eH	r60 H	r120 H
eH	eH	r60H	r120H
r60 H	r60H	r120H	eH
r120 H	r120H	e H	r60H

3. Suppose that $\phi : G \to G'$ is a group homomorphism.
(a) Prove directly that $K = \ker(\phi)$ is a subgroup of G. (7 points)

Solution: We first note that $\phi(e_G) = e_{G'}$ (as we proved in class) and so $e_G \in K$.

Now we tackle closure. Suppose that $a, b \in K$ which just means that $\phi(a) = e_{G'}$ and $\phi(b) = e_{G'}$. We need to show that $ab \in K$. So $\phi(ab) = \phi(a)\phi(b) = e_{G'}e_{G'} = e_{G'}$ which is exactly what we wanted to show.

Finally, we handle inverses. Suppose that $a \in K$ so that $\phi(a) = e_{G'}$. We need to show that $\phi(a^{-1}) = e_{G'}$ as well. But

$$\phi(a^{-1}) = (\phi(a))^{-1} = e_{G'}^{-1} = e_{G'}$$

which completes the proof.

(b) Prove directly that K is normal. (6 points)

Solution: We need to show that $xKx^{-1} \subseteq K$ for all $x \in G$. Fix an $x \in G$ and $k \in K$, we need to show that $xkx^{-1} \in K$. In other words, we need to show that $\phi(xkx^{-1}) = e_{G'}$. But

$$\phi(xkx^{-1}) = \phi(x)\phi(k)\phi(x^{-1}) = \phi(x)e_{G'}\phi(x^{-1}) = \phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_G) = e_{G'}$$

as desired. This completes the proof.

(c) Suppose now that G is a finite group and that ϕ is surjective. Prove that the order of G' divides the order of G (12 points)
Hint: Use the first homomorphism theorem.

Solution: If K is the kernel of ϕ, it is a subgroup. We know that $|G| = |G/K||K|$ by Lagrange's theorem. On the other hand, the first homomorphism theorem tells us that G/K is isomorphic to G'. But isomorphic groups have the same number of elements because there is a bijection between them. Therefore $|G/K| = |G'|$. Plugging this back into our original equation, we have

$$|G| = |G/K||K| = |G'||K|.$$

This proves that the order of G' divides the order of G as desired.

4. Suppose that G is a group, and that $G/Z(G)$ is cyclic. Prove that G is necessarily Abelian. (20 points)

Solution: For simplicity of notation, we set $Z = Z(G)$. Note first that Z is always a normal subgroup, so the question makes sense. Write $G/Z = \langle aZ \rangle$ since G/Z is cyclic. Now choose $b, c \in G$. We need to prove that $bc = cb$. Because the cosets of Z partition the group G, and those cosets are all of the form $a^i Z$ for some $i \in \mathbb{Z}$, we know that $b \in a^i Z$ and $c \in a^j Z$ for some $i, j \in \mathbb{Z}$. Write $b = a^i z_1$ and $c = a^j z_2$ for some $z_1, z_2 \in Z$. Then

$$bc = (a^i z_1)(a^j z_2) = (a^i a^j)(z_1 z_2) = (a^{i+j})(z_1 z_2) = (a^{j+i})(z_2 z_1) = a^j a^i z_2 z_1 = a^j z_2 a^i z_1 = cb.$$

This repeatedly uses the fact that elements of $Z = Z(G)$ commute with everything.

(EC) Suppose that G is an Abelian group and that N_1, N_2 are subgroups. Suppose additionally that $N_1 N_2 = G$. Consider the group

$$G/N_1 \times G/N_2 = \{(g_1 N_1, g_2 N_2) \mid g_i \in G/N_i\}$$

Here the symbol $\times$ just means ordinary Cartesian product. The multiplication operation on the group is entry-wise[1]. Prove that

$$G/(N_1 \cap N_2) \simeq G/N_1 \times G/N_2$$

by using the first homomorphism theorem. (10 points)

Hint: Write down a map from G to $G/N_1 \times G/N_2$ and show it is surjective. The argument is similar to the Chinese Remainder Theorem.

Solution: Consider the function $\phi : G \to (G/N_1) \times (G/N_2)$ defined by the rule:

$$\phi(g) = (gN_1, gN_2).$$

Note that $\phi(ab) = (abN_1, abN_2) = (aN_1, aN_2) \cdot (bN_1, bN_2) = \phi(a) \cdot \phi(b)$ which proves that ϕ is a homomorphism. We now want to show that ϕ is surjective.

Suppose that $(gN_1, g'N_2) \in G/N_1 \times G/N_2$. By hypothesis, $N_1 N_2 = G$, and so $g = n_1 n_2$ and $g' = n_1' n_2'$ for some $n_1, n_1' \in N_1$ and $n_2, n_2' \in N_2$. Now, consider

$$\phi(n_2 n_1') = (n_2 n_1' N_1, n_2 n_1' N_2) = (n_2 N_1, n_1' n_2 N_2) = (n_2 N_1, n_1' N_2) = (n_2 n_1 N_1, n_1' n_2' N_2) = (gN_1, g' N_2).$$

This proves that ϕ is surjective.

Therefore, we know that $G/\ker(\phi) = G/N_1 \times G/N_2$. In particular, if we can show that $\ker(\phi) = N_1 \cap N_2$, we are done. Now, $x \in \ker(\phi)$ if and only if $\phi(x) = (xN_1, xN_2) = (eN_1, eN_2) = e_{G/N_1 \times G/N_2}$. But this happens if and only if both $xN_1 = eN_1$ and $xN_2 = eN_2$ which is equivalent to $x \in N_1$ and $x \in N_2$. But this last statement is just the same as $x \in N_1 \cap N_2$. In particular, since we did this with if and only if all the way down, we have shown that $\ker(\phi) = N_1 \cap N_2$.

[1] In other words, $(aN_1, bN_2)(cN_1, dN_2) = (acN_1, bdN_2)$.