HOMEWORK # 8
DUE WEDNESDAY MARCH 23RD

MATH 435 SPRING 2011

In this homework assignment, you will use the following definition of a ring.

Definition 0.1. A ring R is a set satisfying the following properties ($a, b, c \in R$ are arbitrary elements).

(0) R has two binary operations denoted by $+$ and $\cdot$.
(1) $a + b = b + a$.
(2) $(a + b) + c = a + (b + c)$.
(3) There is an element $0 \in R$ such that $0 + a = a = a + 0$.
(4) There is an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$.
(5) $a(b + c) = ab + ac$.

Often rings are also assumed to satisfy the following (optional) properties.

(6*) $a(bc) = (ab)c$. (associativity)
(7*) $ab = ba$. (commutativity)
(8*) There is an element $1 \in R$ such that $1a = a = a1$. (identity)

All rings below will be assumed to satisfy (6*), associativity, but not necessarily (7*) or (8*), unless specified.

1. Give an example of a finite noncommutative ring. Give an example of an infinite noncommutative ring that does not have a multiplicative identity.
2. Consider the set $\{0, 2, 4\}$ under addition and multiplication modulo 6. Verify that this is a ring with a multiplicative identity.
3. Give an example of a ring R with elements $a, b \in R$ such that the equation $ax = b$ has more than one solution. Give a different example when it has zero solutions.
4. Show that a ring that is a *cyclic* group under addition is commutative.
5. Suppose that R is a ring and $a, b \in R$. For any integers $m, n \in \mathbb{Z}$, we define ma and nb to be a added to itself m-times respectively, b added to itself n-times. Find an example of distinct integers m, n and elements of a ring a, b such that $ma = mb$ and $na = nb$ but $a \neq b$. Show this can't happen if n and m are relatively prime.
6. Let $n > 1$ be an integer. Suppose that R is a ring such that $x^n = x$ for all $x \in R$. Show that $ab = 0$, for any $a, b \in R$ implies then that $ba = 0$. If n is even, prove additionally that $a = -a$ for all $a \in R$.
7. Show that every element of $\mathbb{Z}_{\text{mod} n}$ is either a zero-divisor or a unit.
8. Suppose that R is a finite commutative ring with multiplicative identity, show that every element of R is either a zero-divisor or a unit.
9. Prove that a finite ring R must have positive characteristic.
10. Suppose that x, y belong to a ring of characteristic p where p is prime. Prove that $(x + y)^p = x^p + y^p$. Give an example to show that the equation does not hold if $p = 4$ (and thus is not prime).
11. Find all units, zero-divisors, and nilpotent elements in $\mathbb{Z}_{\text{ mod } 3} \oplus \mathbb{Z}_{\text{ mod } 6}$ (here the addition and multiplication are both component-wise).
12. Consider the set $\mathbb{Z}_{\text{ mod } n}[i] = \{a + bi | a, b \in \mathbb{Z}_{\text{ mod } n}\}$ with addition and multiplication defined mod n and with the relation $i^2 = -1$. Prove that this is not an integral domain if $n = 2, 5$ or 13 but it is an integral domain if $n = 3, 7$ or 11.