WORKSHEET #3 – MATH 3210
FALL 2018

DUE MONDAY SEPTEMBER 24TH

You may work in groups of up to 4. Only one worksheet is required per group.

We recall a definition.

Definition. Recall that a sequence $\{a_n\}$ is called Cauchy if for every $\epsilon > 0$, there exists some $N > 0$ so that if $m, n > N$, we have that

$$|a_m - a_n| < \epsilon.$$

1. Suppose that $\{a_n\}$ is a sequence such that

$$|a_{n+1} - a_n| < \frac{1}{2^n}$$

for all n. Show that $\{a_n\}$ is Cauchy and hence convergent.

We now briefly discuss $\liminf$ and $\limsup$. You can think of these as the infimum or supremum of the tail of the sequence (respectively), but first we approach it as the book does.

Definition. Suppose $\{a_n\}$ is a sequence. Define $i_n = \inf\{a_k \mid k \geq n\}$ and $s_n = \sup\{a_k \mid k \geq n\}$.

2. Suppose that $\{a_n\}$ is a bounded sequence. Prove carefully that $\{i_n\}$ is also bounded.

3. Now prove that $\{i_n\}$ is non-decreasing. Hence it is convergent.

By symmetry, one can argue that $\{s_n\}$ is non-increasing and bounded. Hence it is also convergent. With this in mind, we define

$$\begin{aligned} \liminf a_n &:= \lim i_n, \\ \limsup a_n &:= \lim s_n. \end{aligned}$$

4. Suppose that $\{a_{n_k}\}$ is a convergent subsequence of a bounded sequence a_n. Prove that $s_{n_k} \geq a_{n_k} \geq i_{n_k}$ and conclude that

$$\limsup a_n \geq \lim a_{n_k} \geq \liminf a_n.$$

Hint: Since i_n converges, any of its subsequences also converge (to the same limit). Likewise with s_n.

See Theorem 2.6.5 for proof that $\limsup a_n$ and $\liminf a_n$ are in fact limits of subsequences of a_n. In other words, they are the biggest possible and smallest possible limits of convergent subsequences.

5. Suppose that $\limsup a_n$ and $\limsup b_n$ are finite. Prove carefully that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

6. Suppose that $f : \mathbb{Z}_{>0} \longrightarrow \mathbb{Q}$ is a bijective function from the integers to the rational numbers. (Such functions exist, google the fact that "the rational numbers are countable" if you are not convinced.) Define a sequence $a_n = f(n)$. Show that for each real number $L \in \mathbb{R}$, there exists a subsequence a_{n_k} of a_n such that

$$\lim_k a_{n_k} = L.$$