WORKSHEET #3 – MATH 3210
FALL 2018

DUE MONDAY SEPTEMBER 24TH

You may work in groups of up to 4. Only one worksheet is required per group.

We recall a definition.

Definition. Recall that a sequence $\{a_n\}$ is called Cauchy if for every $\epsilon > 0$, there exists some $N > 0$ so that if $m, n > N$, we have that

$$|a_m - a_n| < \epsilon.$$

1. Suppose that $\{a_n\}$ is a sequence such that

$$|a_{n+1} - a_n| < \frac{1}{2^n}$$

for all n. Show that $\{a_n\}$ is Cauchy and hence convergent.

Solution: Suppose $\epsilon > 0$ and choose $N = \log_2(1/\epsilon) + 1$. Then suppose that $n, m > N$. Without loss of generality suppose that $n \geq m$. It follows that

$$\begin{array}{rl} & |a_n - a_m| \\ = & |a_n - a_{n-1} + a_{n-1} - \cdots + a_{m+1} - a_m| \\ \leq & |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{m+1} - a_m| \\ \leq & \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^m} \\ = & \frac{1}{2^m}(\frac{1}{2^{n-1-m}} + \frac{1}{2^{n-2-m}} + \cdots + 1/2 + 1) \\ \leq & \frac{1}{2^m}(\frac{1}{1-1/2}) \\ = & \frac{1}{2^m}(2) \\ = & \frac{1}{2^{m-1}} \end{array}$$

where in the above, we are using our hypothesis and the fact that the geometric series $1 + r^2 + r^3 + \cdots = \frac{1}{1-r}$. Now since $m > N = \log_2(1/\epsilon) + 1$, we have that

$$\frac{1}{2^{m-1}} < \frac{1}{2^{N-1}} = \frac{1}{2^{\log_2(1/\epsilon)}} = \frac{1}{1/\epsilon} = \epsilon$$

which proves that $\{a_n\}$ is Cauchy.

We now briefly discuss lim inf and lim sup. You can think of these as the infimum or supremum of the tail of the sequence (respectively), but first we approach it as the book does.

Definition. Suppose $\{a_n\}$ is a sequence. Define $i_n = \inf\{a_k \mid k \geq n\}$ and $s_n = \sup\{a_k \mid k \geq n\}$.

2. Suppose that $\{a_n\}$ is a bounded sequence. Prove carefully that $\{i_n\}$ is also bounded.

Solution: We know that there exists K such that $K \leq a_n$ for all n. Thus $K \leq \inf\{a_n\}$. Hence for any n, we have the weaker statement that $K \leq \inf\{a_k \mid k \geq n\}$, and thus $K \leq i_n$. In particular, i_n is bounded below. Likewise we can find L so that $a_n \leq L$ for all N. Hence $i_n \leq \inf\{a_k \mid k \geq n\} \leq \sup\{a_n\} \leq L$. Thus i_n is bounded above as well.

3. Now prove that $\{i_n\}$ is non-decreasing. Hence it is convergent.

Solution: Note that we have the set containment

$$\{a_k \mid k \geq n+1\} \subseteq \{a_k \mid k \geq n\}$$

and so

$$i_{n+1} = \inf\{a_k \mid k \geq n+1\} \geq \inf\{a_k \mid k \geq n\} = i_n$$

which shows that $\{i_n\}$ is non-decreasing. Note that a bounded above non-decreasing sequence of real numbers is convergent.

By symmetry, one can argue that $\{s_n\}$ is non-increasing and bounded. Hence it is also convergent. With this in mind, we define

$$\begin{aligned} \liminf a_n &:= \lim i_n, \\ \limsup a_n &:= \lim s_n. \end{aligned}$$

4. Suppose that $\{a_{n_k}\}$ is a convergent subsequence of a bounded sequence a_n. Prove that $s_{n_k} \geq a_{n_k} \geq i_{n_k}$ and conclude that

$$\limsup a_n \geq \lim a_{n_k} \geq \liminf a_n.$$

Hint: Since i_n converges, any of its subsequences also converge (to the same limit). Likewise with s_n.

Solution: Since $i_n = \inf\{a_k \mid k \geq n+1\}$, we see that $i_n \leq a_n$. Likewise we see that $a_n \leq s_n$. Hence if we take the same subsequence of i_n, a_n, s_n we still have

$$i_{n_k} \leq a_{n_k} \leq s_{n_k}.$$

Since subsequences of convergent sequences converge (to the same thing), we have, taking limits, that

$$\liminf a_n = \lim i_{n_k} \leq \lim a_{n_k} \leq s_{n_k} = \limsup a_n$$

as desired.

See Theorem 2.6.5 for proof that $\limsup a_n$ and $\liminf a_n$ are in fact limits of subsequences of a_n. In other words, they are the biggest possible and smallest possible limits of convergent subsequences.

5. Suppose that $\limsup a_n$ and $\limsup b_n$ are finite. Prove carefully that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

Solution: For any two sets A and B, recall that $A + B = \{a + b \mid a \in A, b \in B\}$. Now, by Theorem 1.5.7(c) in the text, we know that

$$\sup(A + B) = \sup(A) + \sup(B).$$

We consider the sets $A_n = \{a_k \mid k \geq n\}$ and $B_n = \{a_k \mid k \geq n\}$.

$$\{a_k + b_k \mid k \geq n\} \supseteq A_n + B_n$$

(note it's really note going to be equal either, the right side contains terms like $a_3 + b_7$, the left side does not). Hence, taking supremums we see that

$$\sup\{a_k + b_k \mid k \geq n\} \supseteq \sup(A_n + B_n) = \sup A_n + B_n.$$

Taking limits produces

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

as desired.

6. Suppose that $f : \mathbb{Z}_{>0} \longrightarrow \mathbb{Q}$ is a bijective function from the integers to the rational numbers. (Such functions exist, google the fact that "the rational numbers are countable" if you are not convinced.) Define a sequence $a_n = f(n)$. Show that for each real number $L \in \mathbb{R}$, there exists a subsequence a_{n_k} of a_n such that

$$\lim_k a_{n_k} = L.$$

Solution: Note the sequence $\{a_n\}$ is already given to us. We need to choose a subsequence. Now, for any $\epsilon > 0$ (for example, for $\epsilon = 1/k$), we have that $(L - 1/k, L + 1/k)$ contains infinitely many rational numbers.

We define our sequence a_{n_k} recursively. First n_1. We choose n_1 so that $a_{n_1} = f(n_1) \in (L - 1, L + 1)$ (there are infinitely many choices that work since all rational numbers are hit).

Now n_2. Since $(L - 1/2, L + 1/2)$ also has infinitely many rational numbers, we can find a number $n_2 \geq n_1$ so that $a_{n_2} = f(n_2) \in (L - 1/2, L + 1/2)$ (note the condition that $n_2 \geq n_1$ is important).

Now for n_k. Suppose we have chosen $n_1 \leq n_2 \leq \cdots \leq n_{k-1}$. Since $(L - 1/k, L + 1/k)$ has infinitely many rational numbers, we can find $n_k \geq n_{k-1}$ so that $a_{n_k} = f(n_k) \in (L - 1/k, L + 1/k)$.

This a_{n_k} is our subsequence. Now, choose $\epsilon > 0$ and fix $K = 1/\epsilon$. Suppose $k \geq K$, then

$$a_{n_k} \in (L - 1/k, L + 1/k) \subseteq (L - 1/K, L + 1/K) = (L - \epsilon, L + \epsilon).$$

In other words, $|a_{n_k} - L| < \epsilon$. This completes the proof.