QUIZ #7 – MATH 3210
FALL 2018

1. Let $f(x) = 2x^3$. Show using the definition that
$$\lim_{x \longrightarrow -2} f(x) = -16.$$
(10 points)

Solution: First we do some scratch work (this does not impact the grade). We want $|2x^3+16| < \epsilon$. But $|2x^3 + 16| = 2 \cdot |x^3 + 8| = 2 \cdot |x + 2| \cdot |x^2 - 2x + 4|$. If we set $\delta = 1$, then $x \in (-1, -3)$, and so $|x^2 - 2x + 4|$ must be in $(7, 19)$ (note since x is negative, we are summing 3 positive numbers in $x^2 - 2x + 4$). Thus we are going to need $|x + 2| < \epsilon/38 \leq \epsilon/(2 \cdot |x^2 - 2x + 4|)$.

Now we do the real proof.

Choose $\epsilon > 0$ and set $\delta = \min(1, \epsilon/38)$. Suppose that x is such that $|x - (-2)| = |x + 2| < \delta$. Then $|x + 2| < 1$ and so $x \in (-1, -3)$. It follows that $|x^2 - 2x + 4| \in (7, 19)$. Therefore since $|x + 2| < \delta \leq \epsilon/38 \leq \epsilon/(2 \cdot |x^2 - 2x + 4|)$ and so
$$|f(x) - 2(-2)^3| = |2x^3 + 16| = 2 \cdot |x + 2| \cdot |x^2 - 2x + 4| < \epsilon$$
and the limit exists and is equal to -16 as claimed.

In terms of grading, give them $+1$ for writing something like "set $\epsilon > 0$". Give them $+4$ points if they make a reasonable choice of delta that *does not involve* x. If they involve x, give them at most 3 points out of 10. The remaining points are at your discretion.

2. Suppose that $I = (a, b)$ is an open interval and $c \in I$. Suppose that $f : I \setminus \{c\} \longrightarrow \mathbb{R}$ is a function such that

$$\lim_{x \longrightarrow c^+} f(x) = L \text{ and } \lim_{x \longrightarrow c^-} = L$$

for some $L \in \mathbb{R}$. Prove using the definition that

$$\lim_{x \longrightarrow c} f(x) = L.$$

(10 points)

Solution: Choose $\epsilon > 0$. Since $\lim_{x \longrightarrow c^+} f(x) = L$, there exists a $\delta_1 > 0$ so that if $x \in (c, c + \delta_1)$, then $|f(x) - L| < \epsilon$. Likewise since $\lim_{x \longrightarrow c^-} f(x) = L$, there exists a $\delta_2 > 0$ so that if $x \in (c - \delta_2, c)$ then $|f(x) - L| < \epsilon$. Set $\delta = \min(\delta_1, \delta_2)$. Suppose that $x \neq c$ and $|x - c| < \delta$. Then either $x > c$ or $x < c$. If $x > c$ then $x \in (c, c + \delta) \subseteq (c, c + \delta_1)$ and so $|f(x) - L| < \epsilon$. Likewise if $x < c$ we have that $x \in (c - \delta, c) \subseteq (c - \delta_2, c)$ and so again $|f(x) - L| < \epsilon$. Either way, we have $|f(x) - L| < \epsilon$ and so

$$\lim_{x \longrightarrow c} f(x) = L$$

as desired.

Give them +1 point for writing something like "choose $\epsilon > 0$". Give +4 points for realizing their are two potentially different δs. Another +2 for taking the minimum. The remaining +3 are at your discretion.