QUIZ #6 – MATH 3210
FALL 2018

1. Answer the following questions. (2 points each)

(a) Give a precise definition of what it means that a function $f : D \longrightarrow \mathbb{R}$ is continuous at some $a \in D$.

Solution: It means for every $\epsilon > 0$ there exists some $\delta > 0$ so that if $|x - a| < \delta$ and $x \in D$, then $|f(x) - f(a)| < \epsilon$.

(b) What is the intermediate value theorem?

Solution: Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function and that y is between $f(a)$ and $f(b)$. Then there exists some $x \in [a, b]$ so that $f(x) = y$.

(c) State precisely the Archimedean property of the real numbers.

Solution: It says for every real number $x \in \mathbb{R}$, there exists natural number N so that $N > x$.

It is also ok to say that for every real number $x > 0$, there exists a natural number N so that $1/N < x$.

(d) State the triangle inequality for numbers $a, b \in \mathbb{R}$.

Solution: $|a + b| \leq |a| + |b|$.

(e) Suppose that $\{a_n\}$ is a bounded sequence. Give a precise definition of $\limsup a_n$.

Solution: For each integer n, define $s_n = \sup\{a_k \mid k \geq n\}$. Then $\limsup a_n$ is defined to be

$$\lim s_n.$$

It would also be correct to say it is the maximal limit of a convergent subsequence of $\{a_n\}$.

2. Suppose that $f : D_f \longrightarrow \mathbb{R}$ is a continuous function and $g : D_g \longrightarrow \mathbb{R}$ is a continuous function. Set $D_{f\circ g} = \{x \in D_g \mid g(x) \in D_f\}$. We have the composition

$$f \circ g : D_{f\circ g} \longrightarrow \mathbb{R}.$$

Prove carefully that $f \circ g$ is continuous. You may use any theorems from the book (except this one).

Solution: Here are two different solutions.

Option #1. Fix a point $a \in D_{f\circ g}$, we will show $f \circ g$ is continuous at a. Suppose that $a_n \longrightarrow a$ is a convergent sequence in $D_{f\circ g}$. It is sufficient to show that we obtain a convergent sequence.

$$f(g(a_n)) \longrightarrow f(g(a))$$

Now, we have the convergence $g(a_n) \longrightarrow g(a)$ since $a_n \longrightarrow a$ is a sequence in D_g and g is continuous. Then $f(g(a_n)) \longrightarrow f(g(a))$ is also convergent by the same logic since f is continuous.

Option #2. Choose $a \in D_{f\circ g}$, we will prove that $f \circ g$ is continuous at a from the definition. Choose $\epsilon > 0$. Since f is continuous at $g(a)$, there exists a $\delta > 0$ so that if $|y - g(a)| < \delta$ and $y \in D_f$, then $|f(y) - f(g(a))| < \epsilon$. Now, letting δ fill the roll of ϵ for the continuous function g, we have that there exists a $\gamma > 0$ so that if $|x - a| < \gamma$, then $|g(x) - g(a)| < \delta$. But then using what we'd already shown setting $y = g(x)$, we have that

$$|f(g(x)) - f(g(a))| < \epsilon.$$