QUIZ #2 – MATH 3210
FALL 2018

1. Consider the set $A = \{-3\} \cup (-1, 2]$ and the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$.

(a) Compute $f(A)$. (4 points)

Solution: We observe that $f(-3) = 9$ and $f((-1, 2]) = [0, 4]$. Hence $f(A) = \{9\} \cup [0, 4]$.

(b) Compute $f^{-1}(A)$. (4 points)

Solution: This time $f^{-1}(\{-3\}) = \emptyset$. $f^{-1}((-1, 2]) = [-\sqrt{2}, \sqrt{2}]$. Hence $f^{-1}(A) = [-\sqrt{2}, \sqrt{2}]$.

Recall the following axioms for an ordered field F and arbitrary $x, y, z \in F$.

- A1 $x + y = y + x$.
- A2 $x + (y + z) = (x + y) + z$.
- A3 $\exists\ 0 \in F$ such that $0 + x = x$.
- A4 For each $x \in F$, $\exists -x \in F$ with $x + (-x) = 0$.
- M1 $xy = yx$.
- M2 $x(yz) = (xy)z$.
- M3 $\exists 1 \in F$ such that $1 \neq 0$ and $1x = x$.
- D $x(y + z) = xy + xz$.
- F If $x \neq 0$, then $\exists x^{-1} \in F$ so that $xx^{-1} = 1$.
- O1 Either $x \leq y$ or $y \leq x$.
- O2 If $x \leq y$ and $y \leq x$ then $x = y$.
- O3 If $x \leq y$ and $y \leq z$ then $x \leq z$.
- O4 If $x \leq y$ then $x + z \leq y + z$.
- O5 If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

2. Prove that if $x \leq y$ and then $-y \leq -x$ using only the axioms above. Please use complete sentences in your justification. (6 points)

Hint: You aren't allowed to multiply by -1 and flip inequalities, use O4 instead.

Solution: By A4, there exists $-x$ and $-y$. We add both of these to the inequality $x \leq y$ and get $x + (-x) + (-y) \leq y + (-x) + (-y)$ by O4. Now we reorganize the two sides to get

$$-y = 0 + (-y) = x + (-x) + (-y) \leq y + (-x) + (-y) = y + (-y) + (-x) = 0 + (-x) = -x$$

where we used properties A3 and and A4 on both sides of the inequality and also used A1 (and implicitly A2) on the right side. But this just says $-y \leq -x$, which is what we wanted.

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function $f(x) = 2x$ and let $A = \{2, 7\} \cup (0, 5]$. Find the greatest lower bound and least upper bound of the subset

$$f(A) \subseteq \mathbb{R}.$$

Solution: We first compute $f(A)$. First $f(\{2, 7\}) = \{4, 14\}$ while $f((0, 5]) = (0, 10]$. Hence

$$f(A) = \{4, 14\} \cup (0, 10] = (0, 10] \cup \{14\}.$$

The greatest lower bound is 0 and the least upper bound is 14.