QUIZ #11 – MATH 3210
FALL 2018

1. Short answer questions (2 points each)

(a) What does $\sum_{k=2}^{\infty}(1/3)^k$ converge to?

Solution: If the sum was from 0 to ∞, we'd have $\frac{1}{1-(1/3)} = 3/2$. However, we are missing the terms $1 + 1/3 = 4/3$. Thus our answer is $3/2 - 4/3 = 1/6$. Alternately, we could just take our sum and write it as $(1/9) \cdot \sum_{k=2}^{\infty}(1/3)^k = (1/9) \cdot (3/2) = 1/6$.

(b) State the first fundamental theorem of calculus.

Solution: Suppose $[a, b]$ is a closed bounded interval and f is a function continuous on $[a, b]$, differentiable on (a, b) with f' integrable on $[a, b]$. Then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

In they leave off that f' is integrable take off a point (but if they instead say f' is continuous, that's ok).

(c) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable with bounded derivative. Is it always integrable on $[0, 1]$?

Solution: Yes. It is continuous, the fact that the derivative is bounded is a red herring.

(d) Compute $\lim_{x\longrightarrow 0} \frac{\sin(x)-x}{x^3}$.

Solution: Since the numerator and denominator are zero, we can use L'Hôpital's rule. Then we need to compute $\lim_{x\longrightarrow 0} \frac{\cos(x)-1}{3x^2}$. The numerator and denominator are still zero, and so we need to compute $\lim_{x\longrightarrow 0} \frac{-\sin(x)}{6x}$. Finally, we apply L'Hôpital's rule again to get $\frac{-1}{6}$.

(e) Suppose $f : (a, b) \longrightarrow \mathbb{R}$ is a function and $c \in (a, b)$. Give a precise definition of the statement that $\lim_{x\longrightarrow c} f(x) = -\infty$.

Solution: It means for every $K \in \mathbb{R}$, there exists $\delta 0$ so that if $|x - c| < \delta$ and $x \in (a, b)$, then $f(x) < K$.

Take off half a point if they forget the $x \in (a, b)$ hypothesis.

2. Consider the function $f : (0,\infty) \longrightarrow \mathbb{R}$ defined by $f(x) = \int_1^x \frac{1}{t}dt$. Directly show the following.

(a) For $a, b \in (0,\infty)$, $f(ab) = f(a) + f(b)$. (5 points)

Hint: Use the chain rule and the second fundamental theorem of calculus to compute the derivative of $f(ax)$. Compare that it to the derivative of $f(x)$. Finally, try plugging in $x = b$ to evaluate a constant. Make sure to use complete sentences.

Solution: We know by the chain rule and fundamental theorem of calculus that $(f(ax))' = a \cdot \frac{1}{xa} = \frac{1}{x} = f'(x)$. Thus $f(ax) = f(x) + C$ for some constant C, plugging in $x = 1$ we get that $f(a) = f(1) + C = 0 + C = C$. Hence $f(ax) = f(x) + f(a)$ and plugging in $x = b$ completes the proof.

Use your judgement in terms of points.

(b) Let $g(x) : \mathbb{R} \longrightarrow (0,\infty)$ be the inverse function of the one from part (a). Prove that $g'(x) = g(x)$. (5 points)

Hint: We derived a formula for the derivative of the inverse function. If you don't remember that formula, you can re-derive it using the identity $x = f(g(x))$.

Solution: We first recall that $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. Since $f'(x) = 1/x$, we see that $(f^{-1})'(y) = \frac{1}{1/f^{-1}(y)} = f^{-1}(y)$ as desired.

Use your judgement in terms of points.