WORKSHEET #5 – MATH 311W

OCTOBER 24TH, 2012
DUE MONDAY, OCTOBER 29TH

We will study relations. First we recall some terminology. Fix a relation $\sim$ on a set X. The elements x, y, z are elements of X.

(a) $\sim$ is called *reflexive* if $x \sim x$ for all $x \in X$.
(b) $\sim$ is called *symmetric* if whenever $x \sim y$ then $y \sim x$.
(c) $\sim$ is called *transitive* if whenever $x \sim y$ and $y \sim z$ then $x \sim z$.
(d) $\sim$ is called *antisymmetric* if whenever $x \sim y$ then $y \not\sim x$.
(e) $\sim$ is called *weakly antisymmetric* if whenever $x \sim y$ and $y \sim x$ then $x = y$. (Many references call this "*antisymmetric*")
(f) $\sim$ is called *total* if for any $x, y \in X$ we have either $x \sim y$ or $y \sim x$ (or both).

A relation is called a *partial order* if it satisfies (a) reflexive, (c) transitive and (e) weakly antisymmetric. A relation is called a *strict partial order* if it is (d) antisymmetric and (c) transitive. Finally, a relation is called a *total order* if it is (c) transitive, (e) weakly antisymmetric, and (f) total.

1. Consider the empty relation $\sim = \emptyset$ on X. Which of the properties (a)-(e) are satisfied. Is it a partial order or a strict partial order or neither?

Solution: It is not reflexive (unless $X = \emptyset$ too) since nothing is related to anything. It is symmetric since $x \sim y$ never happens (think unicorns). Likewise it is transitive and antisymmetric and weakly antisymmetry. It is not total (as long as X is not empty).

2. Same question as **1.** but now assume that $\sim = X \times X$.

Solution: Everything is related to everything, so it is reflexive. It is also symmetric, transitive. It is definitely not antisymmetric or weakly antisymmetric (unless X is very small) but it is definitely total.

3. Prove that every antisymmetric relation is weakly antisymmetric.

Solution: Since $x \sim y$ and $y \sim x$ never happens, it is weakly antisymmetric.

4. Consider $X = \mathbb{Z}$. Write $x \sim y$ if $x|y$. What properties (a) – (f) does this satisfy? Is it a partial, strict partial, or total order?

Solution: Since x divides itself, it is reflexive. It's not symmetric though. It is transitive since if $x|y$ and $y|z$ then $x|z$ certainly. It's not antisymmetric. Furthermore, it is not weakly anti symmetric because $-7|7$ and $7|-7$ but $7 \neq -7$ (if I had chosen $X = \mathbb{Z}_{\geq 0}$, it would have been weakly antisymmetric). It's not total since 7 does not divide 3 and 3 does not divide 7.

5. Let $X = \mathbb{R} \times \mathbb{R}$. Define a relation on X by saying that $(a, b) \sim (c, d)$ if $a \geq c$ and $b \geq d$. Prove that $\sim$ is a partial order but not a total order.

Solution: Certainly $(a, b) \sim (a, b)$ since $a \leq a$ and $b \geq b$. Therefore the relation is reflexive. Suppose now that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Thus $a \geq c$ and $b \geq d$ and also $c \geq e$ and $d \geq f$. Therefore $a \geq e$ and $b \geq f$. Thus $(a, b) \sim (e, f)$ and so the relation is transitive. Finally, suppose both that $(a, b) \sim (c, d)$ and $(c, d) \sim (a, b)$. Thus $a \geq c$ and $c \geq a$ so $a = c$. Likewise $b = d$. Thus $(a, b) = (c, d)$ and so $\sim$ is a partial order.

To prove it is not total consider the two elements $(0, 2)$ and $(3, 1)$. Note that $(0, 2) \not\sim (3, 1)$ since $0 < 3$ and also that $(3, 1) \not\sim (0, 2)$ since $1 < 2$.