WORKSHEET #7 – MATH 2200
SPRING 2018

NOT DUE

This should help you prepare for the final.

1. Short answer questions.

(a) Define what it means for a function $f : A \longrightarrow B$ to be surjective.

(b) Give an example of a function that is not injective (make sure to specify the domain and codomain).

(c) What is $\{1, 2, 3, 23\} \cap \{3, 2, 1\}$?

(d) What is $\varphi(75)$ where φ is Euler's phi function

(e) Suppose $P(x, y)$, $Q(x), R(y)$ are logical propositions. Rewrite the statement

$$\neg(\forall x, \exists y, (P(x, y) \longrightarrow (Q(x) \vee \neg R(y))))$$

so that no negation symbol appears outside a logical operator ($\longrightarrow$, $\vee$, $\wedge$).

(f) Consider the function $f(x) = 3^x + x^3$. True or false, $f(x)$ is $O(2^x)$.

(g) Consider the system of congruences $x \equiv_5 2$ and $x \equiv_7 3$. How many solutions does this system have between 0 and $104 = (35 \cdot 3) - 1$?

(h) Compute $\gcd(35, 55)$.

(i) Give an example of a relation that is not an equivalence relation.

(j) Give an example of a function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ that is injective but not surjective.

(k) Write the number 33 in base 7.

(l) Consider the set $S = \{-5, -2, -1, 2, 3, 5, 6\}$. Define an equivalence relation on S as $x \sim y$ if $x^2 = y^2$ (you don't need to show this is an equivalence relation). Compute $[2]$, the equivalence class of 2.

(m) Is the open interval $(2, 4) = \{x \in \mathbb{R} \mid 2 < x < 4\}$ finite, countably infinite, or uncountable?

(n) Briefly describe the all comparisons that would be done when doing a binary search on the list $\{1, 3, 4, 5, 7, 8, 9, 9.5, 10\}$, searching for the number 4.

(o) Find an integer n so that the statement $2^{n-1} \equiv_n 1$ is false.

(p) Rewrite the following statement using (nested) quantifiers and logical propositions.

For every problem on the final, there exists some student in our class who will get full credit on that problem.

(q) What does the pigeonhole principal say?

(r) If I have N socks, how many ways are there to choose 3 socks (where order doesn't matter)? What about if order does matter?

(s) Is it true that $-8 \equiv_6 16$?

(t) Is the statement $\neg p \implies (\neg p \vee q)$ a tautology?

(u) What is the inverse of 3 modulo 7?

(v) Suppose $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions. Which of the following notations make sense: $f \circ g$ or $g \circ f$.

2. Run the extended Euclidean algorithm on the integers 63 and 77 and use it to find integers s and t such that $s \cdot 63 + t \cdot 77 = 7$. Make sure to explain each step.

3. Solve the system of congruences

$$\begin{aligned} x &\equiv_7 0 \\ 2x &\equiv_5 1 \\ x &\equiv_4 2 \end{aligned}$$

4. Suppose $S = \{1, 2, 3\}$. Compute $\mathcal{P}(S) \cap \{\emptyset, 1, 2, \{3\}, S\}$.

5. Write down a bijective function between $\mathbb{Z}_{>0}$ and $S = \{x \in \mathbb{Z} \mid x \text{ is odd } \}$ (the odd integers). Prove carefully that your function is bijective.

6. Prove carefully that if $\phi : U \longrightarrow V$ is injective and $\psi : V \longrightarrow W$ is injective, that $\psi \circ \phi$ is also injective.

7. Prove carefully by induction that $9^n - 1$ is divisible by 8 for all integers $n \geq 1$.

8. Use the well ordering principal to give a careful proof of the fact that

$$\gcd(a, b) = \min\{sa + tb > 0 \mid s, t \in \mathbb{Z}\}.$$

9. Write down a recursive algorithm (pseudo-code is fine) which computes the gcd of two numbers a and b.

10. Suppose that A and B are countable infinite sets such that $A \cap B = \emptyset$. Prove that $A \cup B$ and $A \times B$ are also countably infinite.

11. Suppose that S is the set of current math majors at the University of Utah. We define a relation $\sim$ on S where $a \sim b$ if a and b got the same grade in Math 2200. Is $\sim$ an equivalence relation? If so, what is the equivalence class of Alice who got an `A` in 2200?

12. Suppose that $S = \{f : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}\}$ is the set of functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$. Define a relation on S where $f \sim g$ if $|f(x) - g(x)| \leq 1$ for all $x \in \{1, 2, 3\}$. Is $\sim$ an equivalence relation? How many functions are related to the constant function $h(x) = 3$?