WORKSHEET #9 – MATH 1260
FALL 2014

DUE, TUESDAY DECEMBER 2ND

Our goal is to understand connectivity. We start with some definitions.

Definition. Suppose $W \subseteq \mathbb{R}^n$ is a set. A subset $A \subseteq W$ is called *an open subset of* W if $A = W \cap U$ for some open set $U \subseteq \mathbb{R}^n$.

Note that A can be an open subset of W even though A is *not* an open subset of $\mathbb{R}^n$.

Definition. Suppose $W \subseteq \mathbb{R}^n$ is a set. A subset $B \subseteq W$ is called *a closed subset of* W if $B = W \cap V$ for some closed set $V \subseteq \mathbb{R}^n$.

Note that B can be a closed subset of W even though B is *not* a closed subset of $\mathbb{R}^n$.

Definition. A set $W \subseteq \mathbb{R}^n$ is called *disconnected* if there is a subset $T \subseteq W$ where $T \neq W$ and $T \neq \emptyset$ and T is both an open subset of W and a closed subset of W. The two subsets, T and $W \setminus T$, of W, are called *a disconnection of* W. Finally, we say that W is *connected* if it is not disconnected.

Definition. A set $W \subseteq \mathbb{R}^n$ is called *path connected* if for every two points $\vec{u}, \vec{v} \in W$ there is a continuous function $\vec{r} : [0,1] \longrightarrow \mathbb{R}^n$ with $\vec{r}(t) \in W$ for all $t \in [0,1]$ and $\vec{r}(0) = \vec{u}$ and $\vec{r}(1) = \vec{v}$.

1. Suppose that $W \subseteq \mathbb{R}^n$ is such that W is a union of two non-empty sets $W = W_1 \cup W_2$, which are disjoint $W_1 \cap W_2 = \emptyset$, and where $W_1 = W \cap U_1$ for some open set $U_1 \subseteq \mathbb{R}^n$ and $W_2 = W \cap U_2$ for some open set $U_2 \subseteq \mathbb{R}^n$. Show that W is disconnected (the converse holds too but I won't ask you to prove it).

Hint: We need to construct T as in the definition of disconnected. Start with $T = W_1$ and show it is both open and closed as a subset of W.

Solution: Indeed, we note that $T = W_1 = W \cap U$ is open in W. We need to show it is also closed. Let $V = \mathbb{R}^n \setminus U_2$ which is closed. Then $V \cap W$ is closed in W. We need to show that $V \cap W = T$. First suppose that $x \in T$, so $x \in W_1$. So $x \in W$ and $x \in W_1$ so since $W_1 \cap W_2 = \emptyset$, we see that $x \notin W_2$. It follows then that $x \notin U_2$ (if it was in U_2, it would be in $U_2 \cap W = W_2$). Since $x \notin U_2$, $x \in V$ so $x \in V \cap W$. Thus $T \subseteq V \cap W$.

For the reverse direction suppose that $x \in V \cap W$. So $x \in V$, thus $x \notin U_2$ and thus $x \in U_1 \cap W = W_1 = T$ (for this last bit, since $x \notin U_2 \cap W = W_2$, we see that $x \in W_1$ since $W = W_1 \cup W_2$. Thus $V \cap W \subseteq T$ and so T is both open and closed in W.

Finally note that $T \neq \emptyset$ since $W_1 \neq \emptyset$ and $T \neq W$ since $W_1 \neq W$.

2. Next suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous. If $W \subseteq \mathbb{R}^n$ is connected, show that $f(W)$ is also connected.

Hint: Suppose instead that $f(W)$ is disconnected and shoot for a contradiction. Choose a nonempty $T \subsetneq f(W)$ that is both open and closed in $f(W)$. Derive a contradiction.

Solution: First let's prove a lemma.

Lemma. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $W \subseteq \mathbb{R}^n$. Let $g : W \longrightarrow f(W)$ be the restriction of f to W. Then if $U \subseteq f(W) = g(W)$ is open in $g(W)$, then $g^{-1}(U)$ is open in W. Likewise if $V \subseteq f(W)$ is closed in $f(W)$, then $g^{-1}(V)$ is closed in W.

Proof. Suppose that $U \subseteq f(W)$ is open. Then $U = U' \cap f(W)$ for some actually open $U' \subseteq \mathbb{R}^m$. We want to show that $f^{-1}(U') \cap W = g^{-1}(U)$. Choose $x \in g^{-1}(U)$, then $f(x) = g(x) \in U \subseteq U'$ so $x \in f^{-1}(U')$ and of course $x \in W$ (since that's the domain of g). Hence $x \in f^{-1}(U') \cap W$ and thus $g^{-1}(U) \subseteq f^{-1}(U') \cap W$.

For the reverse direction, suppose that $x \in f^{-1}(U') \cap W$. Then $g(x) = f(x) \in U'$. But $g(x) \in f(W)$ as well so that $g(x) \in U' \cap f(W) = U$. Thus $x \in g^{-1}(U)$ so that $f^{-1}(U') \cap W \subseteq g^{-1}(U)$.

The second part follows from the following observation. If $V \subseteq f(W)$ is closed in W, then $f(W) \setminus V$ is open in $f(W)$ (as we saw in class) and thus $g^{-1}(f(W) \setminus V)$ is open. But $g^{-1}(f(W) \setminus V) = W \setminus g^{-1}(V)$ (to see this $x \in g^{-1}(f(W) \setminus V)$ is equivalent that $g(x) \notin V$ and $x \in W$. That's equivalent to $x \in g^{-1}(V)$ and $x \in W$, or $x \in g^{-1}(V) \cap W$). $\square$

The rest is really easy. If $T \subseteq f(W)$ is is both open and closed in $f(W)$ and $T \neq \emptyset$ and $T \neq f(W)$. Let $g : W \longrightarrow f(W)$ be the restriction of f. Then $g^{-1}(T)$ is open and closed in W. Since $T \neq f(W)$, $g^{-1}(T)$ is not equal to W. Since $T \neq \emptyset$, $g^{-1}(T)$ is not the emptyset. Thus $g^{-1}(T)$ proves that W is not connected.

It can be difficult to show that a given set is connected because you have to prove it is not disconnected (ie, prove a negative). Let us take on faith that $[0, 1]$ is connected (or look up how to do it).

3. Suppose that $W \subseteq \mathbb{R}^n$ is path connected. Show that W is also connected.

Hint: Suppose that W is disconnected and let T be as in the definition above. Choose $\vec{u} \in T$ and choose $\vec{v} \in W \setminus T$. Let $\vec{r}$ be a path between $\vec{u}$ and $\vec{v}$. Then consider $\vec{r}^{-1}(T)$ and derive a contradiction.

Solution: Suppose that $U \subseteq W$ is both open and closed and both nonempty and not equal to W. Choose $\vec{u} \in U$ and $\vec{v} \notin U$. Let $g : [0, 1] \longrightarrow W$ be continuous such that $g(0) = \vec{u}$ and $g(1) = \vec{v}$ which exist since W is path connected. Then $g([0, 1])$ is connected by **2**. But $U \cap g([0, 1])$ is both open and closed in $g([0, 1])$ and is not empty (since it contains $\vec{u} = g(0)$ and not equal to $g([0, 1])$ since it doesn't contain $\vec{v} = g(1)$.

4. Suppose that $W \subseteq \mathbb{R}^n$ is a connected and compact subset of $\mathbb{R}^n$. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function. Prove that $f(W)$ is a closed interval.

Hint: You know $f(W)$ is closed and bounded and also connected. Use this.

Solution: We know that $f(W)$ is closed and bounded so it contains a maximum element b and a minimum element a. So $f(W) \subseteq [a, b]$. If $c \in [a, b]$ is not in $f(W)$, then $T = f(W) \cap (c, \infty)$ disconnects $f(W)$.

5. There is a difference between connected sets and path connected sets. Consider the following "comb like space".

Let $W \subseteq \mathbb{R}^2$ be the union of the following sets.

(1) First start with $(0, 1] \times \{0\} = \{(a, 0) \mid a \in (0, 1]\}$.

(2) Next consider the teeth of the comb $\{1/i\} \times [0, 1] = \{(1/i, b) \mid b \in [0, 1]\}$ for every integer $i > 0$. (there are infinitely many teeth)

(3) Add one more tooth to the comb $\{0\} \times (0, 1] = \{(0, b) \mid b \in (0, 1]\}$.

Draw this set (note that the origin is in it), and then write a heuristic argument explaining why it is not path connected but why it is connected.

Solution: I won't draw it, but the reason it is not path connected is there is no way to use a curve to connect the point $\langle 0, 0.5\rangle$ with say $\langle 0.5, 0\rangle$ since $\langle 0, 0\rangle$ is not in W.

W is however connected, since if T is an open and closed set of W containing $\langle 0, 0.5\rangle$, it also contains points on some of the other teeth. These teeth are connected to every other point on the comb not on the first tooth.