WORKSHEET #9 – MATH 1260
FALL 2014

DUE, TUESDAY DECEMBER 2ND

Our goal is to understand connectivity. We start with some definitions.

Definition. Suppose $W \subseteq \mathbb{R}^n$ is a set. A subset $A \subseteq W$ is called *an open subset of* W if $A = W \cap U$ for some open set $U \subseteq \mathbb{R}^n$.

Note that A can be an open subset of W even though A is *not* an open subset of $\mathbb{R}^n$.

Definition. Suppose $W \subseteq \mathbb{R}^n$ is a set. A subset $B \subseteq W$ is called *a closed subset of* W if $B = W \cap V$ for some closed set $V \subseteq \mathbb{R}^n$.

Note that B can be a closed subset of W even though B is *not* a closed subset of $\mathbb{R}^n$.

Definition. A set $W \subseteq \mathbb{R}^n$ is called *disconnected* if there is a subset $T \subseteq W$ where $T \neq W$ and $T \neq \emptyset$ and T is both an open subset of W and a closed subset of W. The two subsets, T and $W \setminus T$, of W, are called *a disconnection of* W. Finally, we say that W is *connected* if it is not disconnected.

Definition. A set $W \subseteq \mathbb{R}^n$ is called *path connected* if for every two points $\vec{u}, \vec{v} \in W$ there is a continuous function $\vec{r} : [0, 1] \longrightarrow \mathbb{R}^n$ with $\vec{r}(t) \in W$ for all $t \in [0, 1]$ and $\vec{r}(0) = \vec{u}$ and $\vec{r}(1) = \vec{v}$.

1. Suppose that $W \subseteq \mathbb{R}^n$ is such that W is a union of two non-empty sets $W = W_1 \cup W_2$, which are disjoint $W_1 \cap W_2 = \emptyset$, and where $W_1 = W \cap U_1$ for some open set $U_1 \subseteq \mathbb{R}^n$ and $W_2 = W \cap U_2$ for some open set $U_2 \subseteq \mathbb{R}^n$. Show that W is disconnected (the converse holds too but I won't ask you to prove it).

Hint: We need to construct T as in the definition of disconnected. Start with $T = W_1$ and show it is both open and closed as a subset of W.

2. Next suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous. If $W \subseteq \mathbb{R}^n$ is connected, show that $f(W)$ is also connected.

Hint: Suppose instead that $f(W)$ is disconnected and choose a nonempty $T \subsetneq f(W)$ that is both open and closed in $f(W)$. Derive a contradiction.

It can be difficult to show that a given set is connected because you have to prove it is not disconnected (ie, prove a negative). Let us take on faith that $[0, 1]$ is connected (or look up how to do it).

3. Suppose that $W \subseteq \mathbb{R}^n$ is path connected. Show that W is also connected.

Hint: Suppose that W is disconnected and let T be as in the definition above. Choose $\vec{u} \in T$ and choose $\vec{v} \in W \setminus T$. Let $\vec{r}$ be a path between $\vec{u}$ and $\vec{v}$. Then consider $\vec{r}^{-1}(T)$ and derive a contradiction.

4. Suppose that $W \subseteq \mathbb{R}^n$ is a connected and compact subset of $\mathbb{R}^n$. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function. Prove that $f(W)$ is a closed interval.

Hint: You know $f(W)$ is closed and bounded and also connected. Use this.

5. There is a difference between connected sets and path connected sets. Consider the following "comb like space".

Let $W \subseteq \mathbb{R}^2$ be the union of the following sets.

(1) First start with $(0,1] \times \{0\} = \{(a,0) \mid a \in (0,1]\}$.
(2) Next consider the teeth of the comb $\{1/i\} \times [0,1] = \{(1/i,b) \mid b \in [0,1]\}$ for every integer $i > 0$. (there are infinitely many teeth)
(3) Add one more tooth to the comb $\{0\} \times (0,1] = \{(0,b) \mid b \in (0,1]\}$.

Draw this set (note that the origin is in it), and then write a heuristic argument explaining why it is not path connected but why it is connected.