WORKSHEET #7 – MATH 1260
FALL 2014

DUE, TUESDAY NOVEMBER 4TH

We are going to prove several of the theorems we have discussed in class recently.

Theorem A *Suppose that $\vec{F} = P\vec{i} + Q\vec{j}$ is a vector field that is continuous on an open connected region $D \subseteq \mathbb{R}^2$. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D then F is a conservative vector field.*

Theorem B *Suppose that $\vec{F} = P\vec{i} + Q\vec{j}$ is a vector field on an open connected simply connected region $D \subseteq \mathbb{R}^2$. Suppose that P and Q have continuous first order derivatives and $P_y = Q_x$. Then $\vec{F}$ is conservative.*

Theorem C (Green's Theorem) *Let C be a positively oriented[1] piecewise smooth[2] simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D then*

$$\int_C P dx + Q dy = \iint_D (Q_x - P_y) dA.$$

Let's first talk about the proof of Theorem A.

1. We want to write $\vec{F} = \nabla f$. Fix a starting point $(a, b) \in D$ and define

$$f(x, y) = \int_C \vec{F} \cdot d\vec{r}$$

where C is any curve in D from (a, b) and (x, y). Explain in a couple sentences why choosing different paths doesn't change the definition of $f(x, y)$.

[1] counter clockwise

[2] It can be broken up into pieces that are defined by differentiable functions

2. We want to show that $f_x = P$ and $f_y = Q$ (this will show that $\vec{F}$ really is a gradient). Let's just show that $f_y = Q$ (the other equality will be similar). Draw a path from (a, b) to (x, y) where the path near (x, y) is purely vertical. Explicitly, say we have a path C_1 from (a, b) to (x, v) and then a path C_2 from (x, v) to (x, y). Draw a picture of this. Write a sentence to explain why you might need to use the fact that D is open if you want to keep your path in D.

3. Again let C_2 be the part of the path from (x, v) to (x, y). Explain why the partial derivative $f_y(x, y)$ is equal to

$$\frac{\partial}{\partial y} \int_{C_2} P dx + Q dy.$$

And then explain why that is equal to

$$\frac{\partial}{\partial y} \int_{v}^{y} P dx + Q dy.$$

4. Finally explain in words why the above is equal to

$$\frac{\partial}{\partial y} \int_{v}^{y} Q dy.$$

Then use the fundamental theorem of calculus to conclude that this is equal to $Q(x, y)$. This finishes our justification (proof) of Theorem A.

We now move on to Theorem C (Green's Theorem). Theorem B will follow from Theorem C and Theorem A.

5. Suppose that our loop C is made up of two curves both going left to right, C_1 defined by $y = f(x)$ on top and C_2 defined by $y = g(x)$ on the bottom with x varying from a to b. In other words $C = C_2 - C_1$ (if we want to keep our orientation counter clockwise, here $-C_1$ means go backwards). Thus D is the region $\{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$. Draw this region.

6. Part of what need to show is that

$$-\iint_D P_y dA = \int_C P dx.$$

Compute $-\iint_D P_y dA$ and also compute $\int_C P dx = -\int_{C_1} P dx + \int_{C_2} P dx$. Show they are equal. (If you want, think about more general Type I regions with vertical line segments over $x = a$ and $x = b$).

7. Suppose now we can prove Green's theorem for regions that are simultaneously type I and type II[3]. Draw a star-shape. Break it up into pieces where each piece is type I and type II. Prove Green's theorem for the star shape as well (this can be tricky, talk to me about it if you are stuck).

Finally we talk about Theorem B. **8.** Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ is such that $P_y = Q_x$ on some open, connected, simply connected region. Show that $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

Hint: Take two non-crossings paths with the same start and end points. Use them to define a region, and conclude that the two paths are equal using Green's theorem.

9. Use Theorem A and **8.** to prove Theorem B.

[3]part **6.** lets us show Green's theorem for type I regions when $Q = 0$, a symmetric argument, switching x and y, lets us show Green's theorem for type II regions when $P = 0$. If you sum up these two results you get Green's theorem for regions that are simultaneously type I and type II