# WORKSHEET #5 – MATH 1260
# FALL 2014

NOT DUE, OCTOBER 7TH

**1.** First we begin with short answer questions.

(a) Are the vectors $\langle 1, 2, 3\rangle$ and $\langle -3, -2, -1\rangle$ perpendicular?

**Solution:** The dot product is not zero, so no.

(b) Find a vector that is perpendicular to $\langle 1, 2, 3\rangle$.

**Solution:** $\langle -3, 0, 1\rangle$ would work, as would $\langle 0, 0, 0\rangle$

(c) True or false, the projection of a vector onto the $xy$-plane is always a unit vector.

**Solution:** False. The vector could already be on the $xy$-plane of any length...

(d) Find the area of the parallelogram defined by the vectors $\langle 1, 2\rangle$ and $\langle -1, 3\rangle$.

**Solution:** Taking the determinant yields $3 - 2(-1) = 5$. So the area is 5.

(e) Find a vector $\vec{w}$ so that if $\vec{u} = \langle 1, 0, -1\rangle$ and $\vec{v} = \langle 0, 0, 2\rangle$, then $\{\vec{u}, \vec{v}, \vec{w}\}$ form a *linearly dependent set.*

**Solution:** I take $\vec{w} = \vec{u}$ so that $\vec{w} = 1\vec{u} + 0\vec{v}$. It's now linearly dependent. You could also take $\vec{w} = \vec{u} + \vec{v}$.

(f) Find a vector $\vec{w}$ so that if $\vec{u} = \langle 1, 0, -1\rangle$ and $\vec{v} = \langle 0, 0, 2\rangle$, then $\{\vec{u}, \vec{v}, \vec{w}\}$ form a *spanning set.*

**Solution:** $\vec{u}$ and $\vec{v}$ already span the $xz$-plane so I can add $\vec{w} = \vec{j} = \langle 0, 1, 0\rangle$.

(g) Setup, but do not evaluate, an integral which computes the arclength of $t \mapsto \langle \cos(t), t\sin(t), t^2\rangle$ for $t$ from 2 to 3.

**Solution:** $\int_2^3 \sqrt{(-\sin(t))^2 + (\sin(t) + t\cos(t))^2 + (2t)^2} dt$

(h) If an ant is climbing down a hill whose height is given by $z = x^2 + y^2 + 3x\cos(y^2)$ and is at position $(1, 0)$, what direction should the ant climb to descend the hill fastest?

**Solution:** $\nabla z = \langle 2x + 3\cos(y^2), 2y - 6xy\sin(y^2)\rangle$. Plugging in $(1, 0)$ gives $\langle 2 + 3, 0\rangle = \langle 5, 0\rangle$. So the ant should move in the opposite direction, towards $\langle -1, 0\rangle = -\vec{i}$.

(i) Find the curvature of the space curve $t \mapsto \langle t, t^2, t^3\rangle$ at the point $\langle 2, 4, 8\rangle$.

**Solution:** This is a little messier. $\vec{r}'(t) = \langle 1, 2t, 3t^2\rangle$ and $\vec{r}''(t) = \langle 0, 2, 6t\rangle$. Since we are interested in the point at $t = 2$, $\vec{r}'(2) = \langle 1, 4, 12\rangle$ and $\vec{r}''(2) = \langle 0, 2, 12\rangle$. We compute the cross product $\vec{r}'(2) \times \vec{r}''(2) = \langle 1, 4, 12\rangle \times \langle 0, 2, 12\rangle = \langle 0, -12, 2\rangle$. The length of this is $\sqrt{144 + 4} = 148^{0.5}$. On the other had $|r'(2)| = \sqrt{1 + 16 + 144} = 161^{0.5}$. Hence the curvature is $\kappa = \frac{148^{0.5}}{161^{1.5}}$.

We continue **1.**

(j) True or false, the normal vector is never a unit vector.

**Solution:** False, it is always a unit vector.

(k) Consider the following integral

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2 + 6)dydx$$

Set it up in polar coordinates (but do not evaluate it).

**Solution:** Note that we are only integrating a half-circle of radius 2. So we setup

$$\int_0^2 \int_{-\pi/2}^{\pi/2} (r^2(\cos(\theta))^2 + r^2(\sin(\theta))^2 + 6)rdrd\theta$$

(l) Compute the cross product $\langle 0, -1, 2\rangle \times \langle 1, 0, 3\rangle$.

**Solution:** $\langle -3, 2, 1\rangle$.

(m) Find the equation of the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, 1, 2)$.

**Solution:** The equation is $z - 2 = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 2(x - 1) + 2(y - 1)$.

(n) Suppose $t \mapsto \vec{r}(t)$ is a parameterization of a space curve. True or false $\vec{r}'(t) \cdot \vec{N}(t) = 0$.

**Solution:** True, we know the $\vec{N}(t)$ is perpendicular to the unit tangent vector $\vec{T}(t)$ which points in the same direction as $\vec{r}'(t)$.

(o) Give an example of a surface $z = f(x, y)$ where

$$\lim_{(x,y)\longrightarrow(0,0)} f(x, y)$$

does not exist but the following do exist:

$$\lim_{x\longrightarrow 0} f(x, 0) \text{ and } \lim_{y\longrightarrow 0} f(0, y).$$

**Solution:** $z = \frac{xy}{x^2+y^2}$

(p) Suppose that we are given a function $f(x, y)$ with $\nabla f(1, 1) = \langle -1, 2\rangle$ describing the height of a hill. Further suppose that the $xy$-coordinates of a person is given by $t \mapsto p(t) = \langle g(t), h(t)\rangle$. If $p(3) = \langle 1, 1\rangle$ and $p'(3) = \langle 0, 1\rangle$, is the person ascending or descending the hill at time $t = 3$?

**Solution:** Using the chain rule, $(f \circ p)'(3) = f_x(1, 1)g'(3) + f_y(1, 1)h'(3)$. Plugging this in we see that $(-1)(0) + (2)(1) = 2$ so he is ascending.

(q) State the second derivative test for finding the maxes or mins of $z = f(x, y)$.

**Solution:** If $(a, b)$ is a point satisfying $f_x(a, b) = f_y(a, b) = 0$ and we set $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$, and the second partial derivatives are continuous then
(1) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a local min at $(a, b)$.
(2) If $D > 0$ and $f_{yy}(a, b) < 0$, then $f$ has a local max at $(a, b)$.
(3) If $D < 0$ then $f(a, b)$ is not a local maximum or minimum (it is a saddle point).

(r) If $\nabla f = \langle 3, 2\rangle$, what is the directional derivative of $f$ in the direction $\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$?

**Solution:** $\langle 3, 2\rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle = \frac{3\sqrt{2}}{4} + \sqrt{2}$

**2.** The base of an aquarium of volume $V$ is made of stone and the sides are glass. If stone costs 5 times as much as glass, what dimensions should the aquarium be (in terms of $V$) in order to minimze the cost of materials. Justify your answer.

**Solution:** We have sides of length $a, b$ and height $h$. The volume is $V = abh$. The cost is $C = 2ah + 2bh + 5ab$ (two sides). We view $V$ as a constant and we try to minimize $C$. If we solve $V = abh$ for $h$ we get $h = V/(ab)$. Plugging this into the cost equation we get $C = 2aV/(ab) + 2bV/(ab) + 5ab = 2V/b + 2V/a + 5ab$. We then try to find local mins and maxes. We take partial derivatives

$$\begin{aligned} C_a = & \ -2V/a^2 + 5b \\ C_b = & \ -2V/b^2 + 5a \end{aligned}$$

These equal zero when $2V = 5a^2b$ and $2V = 5ab^2$. Since obviously we need $a, b > 0$ we have $5a^2b = 5ab^2$ and hence $a = b$ (this makes sense, the base should be a square to minimize cost). Then going back to $C_a = 0 = C_b$ equations we see that $a = (2V/5)^{1/3} = b$ as well. Of course then $h = V/(ab) = V/(2V/5)^{2/3} = \frac{5^{2/3}V^{1/3}}{2^{2/3}}$. This is our critical point. We need to verify that this is a min (it has to be, as obviously making something really long and skinny will have huge costs) but let's use the second derivative test for fun!

$$\begin{aligned} C_{aa} = & \ 4V/a^3 \\ C_{bb} = & \ 4V/b^3 \\ C_{ab} = & \ 5 \end{aligned}$$

and so we write $D = 4V^2/(a^3b^3) - 25$. Plugging in our values for $a$ and $b$ we get

$$D = 4V^2/(2V/5)^2 - 25 = 25 - 25 = 0.$$

so the second derivative test tells us nothing :-( We have to argue by logic as described above.

**3.** Find the local maximums and local minimums of the following surface

$$z = xy + \frac{1}{x} + \frac{1}{y}$$

**Solution:** The partial derivatives are basically the same as those in **2.**. You can work out the details (there will just be a single local min at $(1, 1)$).

**4.** Find the distance of the point $(1, 2, 3)$ from the tangent plane to the surface $z = x^3 + y^3 + xy$ at $(1, 1, 3)$.

**Solution:** First we compute the tangent plane to the surface at the specified point. The partials are $z_x = 3x^2 + y$ and $z_y = 3y^2 + x$. So plugging in $(1, 1)$ and using the tangent plane formula we get

$$z - 3 = 4(x - 1) + 4(y - 1) \text{ or } 4x + 4y - z = 5.$$

For simplicity, let's move the origin to $(1, 1, 3)$. Then our tangent plane becomes $4x + 4y - z = 0$ and the point $(1, 2, 3)$ becomes $(0, 1, 0)$. So we need to find that distance. The normal vector to the tangent plane is $\langle 4, 4, -1 \rangle$. We project $\langle 0, 1, 0 \rangle$ onto that vector and we get

$$\frac{\langle 4, 4, -1 \rangle \cdot \langle 0, 1, 0 \rangle}{|\langle 4, 4, -1 \rangle|} = \frac{4}{\sqrt{17}}.$$

which is slightly smaller than 1 (and that makes sense, because $(1, 2, 3)$ is a distance of 1 from $(1, 1, 3)$.)

**5.** Reparameterize the space curve $t \mapsto \langle 2t + 1, 3t, -t \rangle$ with respect to arc length.

**Solution:** Because this is a parameterization of a line and it is a constant rate, this is really easy (no integrals required). Imagining this as the position of a particle, in one unit of time, the particle moves $\langle 2, 3, -1 \rangle$ which is a vector of length $\sqrt{14}$. Hence dividing $t$ by $\sqrt{14}$ gives us a parameterization

$$s \mapsto \langle (2s/\sqrt{14}) + 1, 3s/\sqrt{14}, -s/\sqrt{14} \rangle.$$

**6.** Find the volume of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y, z = 0, y = 4$.

**7.** Sketch the region of integration of the following integral

$$\int_1^2 \int_0^{\ln x} x dy dx$$

and then rewrite the integral as

$$\int_a^b \int_{g_1(y)}^{g_2(y)} x dx dy.$$

In particular, find the constants $a, b$ and the functions $g_1(y)$ and $g_2(y)$.

**8.** Setup an integral to find the volume of the solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$ and $2x + 2y - z + 10 = 0$.