WORKSHEET #2 – MATH 1260
FALL 2014

DUE MONDAY, SEPTEMBER 8TH

For this assignment, you are allowed and encouraged to work in groups. Each group only has to turn in one assignment worksheet, but make sure it is done neatly.

1. Fix a nonzero vector $\vec{u} = \langle a, b, c \rangle \in \mathbb{R}^2$. Consider all of the vectors with tail at the origin and perpendicular to $\vec{u}$. What do the end points of these vectors look like? Write down an equation in x, y, z to describe this set.

Solution: The set of such vectors form a plane. The equation is $ax + by + cz = 0$, in other words $\vec{u} \cdot \langle x, y, z \rangle = 0$.

Generally a function $ax + by + cz = 0$ defines a plane through the origin and perpendicular to this plane is the *normal vector* $\langle a, b, c \rangle$.

2. Find the equation of a plane perpendicular to $\vec{u} = \langle a, b, c \rangle$ and passing through the point (d, e, f).

Hint: Try shifting the equation we considered above.

Solution: $a(x - d) + b(x - e) + c(x - f) = 0$.

3. Given a direction vector $\vec{v} = \langle x, y, z \rangle$ and starting point $Q = \langle a, b, c \rangle$ consider

$$t\vec{v} + Q = t\langle x, y, z \rangle + \langle a, b, c \rangle = \langle xt + a, yt + b, zt + c \rangle.$$

As the parameter t varies, what points does this expression hit? Describe the answer geometrically.

Hint: If you get stuck, try making up some numbers x, y, z, a, b, c and plugging in $t = 0, 1, 2, \ldots$.

Solution: These points parameterize a line pointing in the direction of $\vec{v}$ and starting at the point Q.

4. What you did in problem **3.** was to consider a *parametric equation of a line*. Consider the line L parameterized by $t \mapsto t\langle 2,4,6\rangle + \langle 0,0,1\rangle$. Find two other parameterizations of the *same* line: one where $t = 0$ gives the same point as the parametrization above and the other where $t = 0$ gives a different point.

Solution: Here are two other parameterizations:

$$t \cdot \langle 1,2,3\rangle + \langle 0,0,1\rangle$$

and

$$t \cdot \langle 2,4,6\rangle + \langle 2,4,7\rangle$$

The first one starts at the same point, it just moves more slowly. The second starts at a different point on the same line (note that $\langle 2,4,7\rangle = \langle 2,4,6\rangle + \langle 0,0,1\rangle$ is on the line because it is the point of the given line when $t = 1$.

5. Consider two planes passing through the origin: $ax + by + cz = 0$ and $dx + ey + fz = 0$. If the planes are different, then they intersect in a line (convince your group of this). Find a parametrization of this line.

Hint: The line will be orthogonal to the normal vectors of both planes, have we learned any way recently to cook up a vector orthogonal to two other vectors?

Solution: The cross product of $\langle a,b,c\rangle$ and $\langle d,e,f\rangle$ is just $(bf - ce)\vec{i} - (af - cd)\vec{j} + (ae - bd)\vec{k}$. So this will be the direction vector. Both planes pass through the origin and hence so does their intersection. Thus our parametrization is:

$$\langle bf - ce, -af + cd, ae - bd\rangle t + \langle 0,0,0\rangle.$$

6. Now consider the following two planes $2x + y - z = 1$ and $x - 3y + 2z = 1$. Are the planes parallel, why or why not? If not, find a parametrization of the line that passes through both of them.

Hint: You can use the same strategy as above to find the direction of the line of their intersection. Then you just need to find a common point on the line.

Solution: The normal vectors of the planes are in different (non-collinear) directions and so the planes are not parallel. The direction vector of the line is again the cross product, and so it is

$$\langle 1 \cdot 2 - (-1) \cdot (-3), (-2) \cdot 2 + (-1) \cdot 1, 2 \cdot (-3) - 1 \cdot 1\rangle = \langle -1,-5,-7\rangle.$$

Next we need to find any point on the line, or in other words, any point on both planes. Solving the first equation for z gives us $z = 2x + y - 1$ and plugging this into the second equation gives us $x - 3y + 4x + 2y - 2 = 1$ and so $5x - y = 3$. I notice that $x = 1$ and $y = 2$ makes that equation true and so then $z = 2 \cdot 1 + 2 - 1 = 3$. We see easily that $\langle 1,2,3\rangle$ is on both planes so we did it right. Now then our parametric formula is:

$$t\langle -1,-5,-7\rangle + \langle 1,2,3\rangle.$$

7. Suppose you are given a plane H defined by $ax + by + cz = 0$ and a vector $\vec{u} = \langle x, y, z \rangle$. Write here what the phrase *the projection of $\vec{u}$ onto H* should mean (draw a pretty picture). Once you have figured that out, write down a general formula for the projection of $\vec{u}$ onto H.

Hint: For the second part, it's easy to project $\vec{u}$ onto the normal vector of the plane. Once you have done that, do some basic vector arithmetic to find the projection of $\vec{u}$ onto H.

Solution: Write $\vec{u} = \vec{v} + \vec{w}$ where $\vec{v}$ is on the plane and $\vec{w}$ is perpendicular to the plane. That is the projection of $\vec{u}$ to the plane (I'd draw a nice picture, but I am typing, sorry).

We first find $\vec{w}$. This is the projection of $\vec{u}$ onto the normal vector of the plane, which is $\langle a, b, c \rangle$ in this case. Hence

$$\vec{w} = \frac{\vec{u} \cdot \langle a, b, c \rangle}{a^2 + b^2 + c^2} \langle a, b, c \rangle.$$

and so

$$\vec{v} = \vec{u} - \frac{\vec{u} \cdot \langle a, b, c \rangle}{a^2 + b^2 + c^2} \langle a, b, c \rangle.$$

8. Where does the parameterized line $t \mapsto t\langle -1, 2, -3 \rangle + \langle 0, 1, 0 \rangle$ intersect the plane $2x + 3y - 4z = 0$?

Solution: The parameterized line is give by $x = -t$, $y = 2t + 1$, $z = -3t$. Plugging these in to the plane equation gives us $2(-t) + 3(2t + 1) - 4(-3t) = 0$. When we simplify we get

$$16t + 3 = 0$$

and so $t = -3/16$. Plugging this back into x, y, z gives us

$$\begin{aligned} x &= 3/16 \\ y &= -6/16 + 1 = 10/16 = 5/8 \\ z &= 9/16 \end{aligned}$$

so the answer is $\langle 3/16, 5/8, 9/16 \rangle$.

9. Consider the two parameterized lines $s \mapsto s\langle 1, 2, 3 \rangle$ and $t \mapsto t\langle 1, 0, -1 \rangle + \langle 0, 4, 8 \rangle$. Do they intersect, and if so, where?

Solution: Setting them equal we get $\langle s, 2s, 3s \rangle = \langle t, 4, -t + 8 \rangle$. Hence $2s = 4$ so $s = 2$ (from the y-coordinate). But the from the x-coordinate we see that $t = s = 2$. Finally we have to check if the z-coordinates match under this hypothesis. Is $3s = 6$ the same as $8 - t = 6$? Yes, and so the intersection point is $\langle 2, 4, 6 \rangle$.

If the two lines where skew (if they did not intersect) then the z-coordinate would *not* have matched.

10. Find the distance of a point $Q = \langle 0, -1, 3 \rangle$ away from the plane $2x + y - z = 2$.

Hint: Parameterize a line starting from Q and perpendicular to the plane. Find where that line intersects the plane.

Solution: Following the hint, consider the line $t\langle 2, 1, -1 \rangle + \langle 0, -1, 3 \rangle$. This means that $x = 2t$, $y = t-1$, $z = -t+3$. Plugging this into the equation of the plane we get $2(2t)+(t-1)-(-t+3) = 2$ and simplifying we get

$$6t - 4 = 2 \text{ or } t = 1$$

Plugging this back into our line formula we obtain

$$x = 2, y = 0, z = 2 \text{ or } \langle 2, 0, 2 \rangle.$$

The distance between this point and Q is

$$\sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$$