WORKSHEET #12 – MATH 1260
FALL 2014

NOT DUE

This is a brief worksheet on tensors. First we have some preliminaries.

1. Suppose that V and W are vector spaces. We define $V \times W$ to be the set of pairs $(\vec{v}, \vec{w})$. We want $V \times W$ to be a vector space. What is the right way to add $(\vec{v}_1, \vec{w}_1) + (\vec{v}_2, \vec{w}_2)$? What is the right way to scale $(\vec{v}, \vec{w})$ by an element $r \in \mathbb{R}$?

We won't use the structure from the previous problem *at all* (it was just to think about)!!!

Suppose that V is a vector space. For any integer $k > 0$ a *k-tensor* is a multi-linear map

$$g : \underbrace{V \times V \times \ldots \times V}_{k\text{-times}} \longrightarrow \mathbb{R}$$

Being multi linear means that

(i) $g(\vec{v}_1, \ldots, \vec{v}_i + \vec{v}_i', \ldots, \vec{v}_k) = g(\vec{v}_1, \ldots, \vec{v}_i, \ldots, \vec{v}_k) + g(\vec{v}_1, \ldots, \vec{v}_i', \ldots, \vec{v}_k)$.

(ii) $g(\vec{v}_1, \ldots, r\vec{v}_i, \ldots, \vec{v}_k) = rg(\vec{v}_1, \ldots, \vec{v}_i, \ldots, \vec{v}_k)$.

2. Let $V = \mathbb{R}^1$. Is the function

$$V \times V \longrightarrow \mathbb{R}$$

which sends $(x_1, x_2) \mapsto x_1 + x_2$ multilinear? In other words, is it a 2-tensor? What about the function $(x_1, x_2) \mapsto x_1 x_2$. Can you find a multilinear function from $V \times V \times V \longrightarrow \mathbb{R}$?

3. Suppose that

$$g : \underbrace{V \times V \times \ldots \times V}_{k\text{-times}} \longrightarrow \mathbb{R} \text{ and } h : \underbrace{V \times V \times \ldots \times V}_{l\text{-times}} \longrightarrow \mathbb{R}$$

are k-tensors and l-tensors respectively. We define a new map

$$g \otimes h : \underbrace{V \times V \times \ldots \times V}_{(k+l)\text{-times}}$$

by

$$(g \otimes h)(\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_{k+l}) = g(\vec{v}_1, \ldots, \vec{v}_k) \cdot h(\vec{v}_{k+1}, \ldots, \vec{v}_{k+l}).$$

Show that $g \otimes h$ is a $(k+l)$-tensor.

4. Consider $V = \mathbb{R}^2$. Consider the function:

$$g : V \times V \longrightarrow \mathbb{R}$$

defined by

$$g(\langle a, b\rangle, \langle c, d\rangle) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that g is a 2-tensor.

A k-tensor ω is called *alternating* if $\omega(\vec{v}_1, \ldots, \vec{v}_i, \ldots, \vec{v}_j, \ldots, \vec{v}_k) = -\omega(\vec{v}_1, \ldots, \vec{v}_j, \ldots, \vec{v}_i, \ldots, \vec{v}_k)$ (ie, switching two vectors flips a sign).

5. Show that the 2-tensor we defined in **4.** is alternating.

6. There is a way to turn a k-tensor into an alternating k-tensor. Let's describe this for 2-tensors. If $g : V \times V \longrightarrow \mathbb{R}$ is 2-tensor, show that

$$\mathrm{alt}(g) : V \times V \longrightarrow \mathbb{R}$$

defined by

$$\mathrm{alt}(g)(\vec{v}_1, \vec{v}_2) = \frac{1}{2}(g(\vec{v}_1, \vec{v}_2) - g(\vec{v}_2, \vec{v}_1))$$

is an alternating 2-tensor. What if g is an alternating 2-tensor, show that $\mathrm{alt}(g) = g$.

7. Can you figure out the formula to turn a 3-tensor into an alternating 3-tensor?