WORKSHEET #11 – MATH 1260
FALL 2014

DUE WEDNESDAY, DECEMBER 10TH

Our goal is to try to understand the *Implicit Function Theorem*. The idea of the implicit function is that we can often show that equations (like $g(\langle x, y\rangle) = 0$) in fact define graphs of functions (at least "locally"). We start by stating the implicit function theorem for $\mathbb{R}^1$.

Theorem A. *Suppose that $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function which is continuously differentiable on an open set U containing $(a, b) \in \mathbb{R}^2$. Suppose further that $g(a, b) = 0$. Suppose that the partial derivative $g_y(a, b) \neq 0$. Then there is an open set $A \subseteq \mathbb{R}$ with $a \in A$ and another open set $B \subseteq \mathbb{R}$ with $b \in B$ with the following property: for each $x \in A$ there is a* unique *$y_x \in B$ with $g(x, y_x) = 0$.*

1. Draw what the theorem says. In particular draw the sets A, B on their appropriate axes and draw the point (a, b). Draw the locus where $g(x, y) = 0$. Also draw the graph of the function $f : A \longrightarrow B$ which is defined by $f(x) = y_x$. Make sure your $g_y(a, b) \neq 0$.

2. Same as **1.** but this time draw your $g(x, y) = 0$ locus in such a way that you really have to make your open sets A and B quite small for the y_x to be unique.

Hint: A spiral might be interesting.

3. What happens if you choose a point (a, b) where $g_y(a, b) = 0$? Draw a picture showing that the theorem can't apply.

Let's prove the implicit function theorem in the special case above. We do this in steps. Hence for problems 4 through ???, we may assume the notation and assumptions of Theorem A.

4. Define $\vec{G} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by the rule $\vec{G}(\langle x, y\rangle) = \langle x, g(x, y)\rangle$. Compute $\det(\mathrm{Jac}_{\vec{G}}(a, b))$, is it zero?

Hint: This is actually *really* easy, just write it down.

5. Since $\det(\mathrm{Jac}_{\vec{G}}(a, b)) \neq 0$, the inverse function theorem[1] says that $\vec{G} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ has an inverse, at least locally near $\langle a, b\rangle$. In particular, we may take W an open set containing $\vec{G}(\langle a, b\rangle)$ and take V an open set containing $\langle a, b\rangle$ with $\vec{G} : V \longrightarrow W$ having an inverse $\vec{G}^{-1}$. Show that $\vec{G}(a, b) = (a, 0)$ and then argue that we may take V to be of the form $A \times B$.

Hint: This is tricky, use the result from the homework saying that images of open sets are open.

[1]**Theorem.** *If $\vec{h} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuously differentiable on an open set U containing $\vec{a}$ and* $\det(\mathrm{Jac}_{\vec{h}}(\vec{a})) \neq 0$, *then there is an open set $V \subseteq \mathbb{R}^n$ containing $\vec{a}$ and an open set $W \subseteq \mathbb{R}^n$ containing $\vec{h}(\vec{a})$ so that $\vec{h} : V \longrightarrow W$ has an inverse $\vec{h}^{-1} : W \longrightarrow V$ which is continuous, differentiable and for all $\vec{y} \in W$ satisfies*

$$(\mathrm{Jac}_{f^{-1}})(\vec{y}) = \left(\mathrm{Jac}_f(f^{-1}(\vec{y}))\right)^{-1}$$

We continue our proof of Theorem A. and in particular keep the notation of the previous problem.

6. Show that $\vec{G}^{-1}(x, y) = \langle x, k(x, y)\rangle$ for some differentiable function $k : W \longrightarrow \mathbb{R}$.

Hint: Remember, $\vec{G}$ itself had a pretty special form.

7. Show that $f(x, k(x, y)) = y$.

8. Finally, going back to the notation of Theorem A, show that we can define $y_x = k(x, 0)$ (explain why this y_x is unique). Why do you need to use the fact that $(a, 0) = \vec{G}(a, b) \in W$?

The general version of the implicit function theorem is as follows.

Theorem B. *Suppose that* $g : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ *is a function which is continuously differentiable on an open set* U *containing* $(\vec{a}, \vec{b}) \in \mathbb{R}^n \times \mathbb{R}^m$ *(here* $\vec{a} \in \mathbb{R}^n$ *and* $\vec{b} \in \mathbb{R}^m$*). Suppose further that* $g(\vec{a}, \vec{b}) = \vec{0} \in \mathbb{R}^m$*. Form the square* $m \times m$ *matrix* M *by taking* m *partial derivatives of* g *in the* $\mathbb{R}^m$*-variables (those that make up* $\vec{b}$*). If* $\det(M(\vec{a}, \vec{b})) \neq 0$ *then there is an open set* $A \subseteq \mathbb{R}^n$ *with* $\vec{a} \in A$ *and another open set* $B \subseteq \mathbb{R}^m$ *with* $\vec{b} \in B$ *with the following property: for each* $\vec{x} \in A$ *there is a* unique $\vec{y}_x \in B$ *with* $g(\vec{x}, \vec{y}_x) = 0$.

9. Draw this theorem in action when $n = 2$ and $m = 1$.

10. Draw this theorem in action when $n = 1$ and $m = 2$.