QUIZ #12– MATH 1260
FALL 2014

11/21/14

1. Short answer, 1 point each.

(a) State stokes theorem.

Solution: Suppose that S in $\mathbb{R}^3$ is a piece-wise smooth surface bounded by a simple closed closed curve C. Suppose further that $\vec{F}$ is a vector field on $\mathbb{R}^3$ all of whose partial derivatives are continuous. Then

$$\int_C \vec{F} \cdot ds = \iint_S (\text{curl}\vec{F}) \cdot dS.$$

(b) Give an example of a vector field $\vec{F} = P\vec{i} + Q\vec{j}$ defined on open region, which satisfies $Q_x - P_y = 0$, but where $\vec{F}$ is not a gradient vector field (in other words, is not conservative).

Solution: Consider the vector field $\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \rangle$. We checked in class that it satisfies $Q_x - P_y = 0$. Note this is not a contradiction because $\vec{F}$ is not defined at the origin and $\mathbb{R}^2 \setminus \{0\}$ is not simply connected.

(c) Define what it means for a region $W \subseteq \mathbb{R}^n$ to be compact.

Solution: It means that every open cover of W has a finite subcover.

(d) Is the following parameterized curve an open subset of $\mathbb{R}^2$, $\vec{r}(t) = \langle t^2, t^3 \rangle$ for $t \in (0, 1)$.

Solution: No, open subsets of $\mathbb{R}^2$ contain open discs / squares around each point of the subset.

(e) State the Heine Borel theorem.

Solution: Closed bounded intervals in $\mathbb{R}$ are compact.

Continued on the back

2. Suppose an extremely complicated surface S (with upward orientation), defined by a (continuously differentiable) equation $z = f(x, y)$ is always ≥ 0 and intersects the $z = 0$ plane in a circle of radius 1 centered at the point $\langle 0, 1, 0\rangle$. Further suppose that the volume of the region above the plane $z = 0$, below the surface $z = f(x, y)$ and above the aforementioned circle, is equal to 2π. Compute

$$\iint_S \langle ye^z, 2y, y+2\rangle \cdot dS.$$

(5 points)

Solution: If S_2 is the circle in the plane (the bottom part of our region), then we have

$$\iint_S \langle ye^z, 2y, y+2\rangle \cdot dS - \iint_{S_2} \langle ye^z, 2y, y+2\rangle \cdot dS = \iiint_E \operatorname{div}(\langle ye^z, 2y, y+2\rangle)dV$$

where we subtract because we orient S_2 upwards and E is the region above. Note $\operatorname{div}(\langle ye^z, 2y, y+2\rangle) = 2$ so

$$\iiint_E \operatorname{div}(\langle ye^z, 2y, y+2\rangle)dV = 2(\text{volume}) = 2(2\pi) = 4\pi.$$

Hence we only need to compute $\iint_{S_2} \langle ye^z, 2y, y+2\rangle \cdot dS$. We parameterize that by $\vec{r}(u, v) = \langle u, v, 2\rangle$ which gives us the normal vector $\langle 0, 0, 1\rangle$ (by the usual computation). Hence we compute

$$\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} (v+2)dudv = \int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} ((v-2)+4)dudv.$$

Now, the function $(v-2)$ integrates to zero over our circle (for obvious geometric reasons). Hence we are just computing

$$\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} 4dudv$$

which comes out to be $4\pi r^2 = 4\pi$. Hence we have

$$\iint_S \langle ye^z, 2y, y+2\rangle \cdot dS = 8\pi.$$