COMPUTER #4 (VECTOR FIELDS ON THE DOUGHNUT) – MATH 1260, FALL 2014

DUE TUESDAY NOVEMBER 18TH

Grading and rubric

Each group should turn in a 2-5 page written report explaining what was done mathematically. It needs to include the following:

(1) The introduction (and introduction to each section) needs to clearly restate the problem to be solved.
(2) Citations and/or restatements of any significant or unusual formulas.
(3) Use graphs and tables where appropriate, or short snippets of Mathematica/Maple/Maxima code.
(4) Include proper spelling and grammar and use of mathematical formulas.
(5) The mathematics needs to be complete and correct.

If that is done, you can receive a 17/20. To receive the 3 final points, you need to do something above and beyond the assignment, in a *mathematical* way. Simply copying a couple paragraphs from the internet will not suffice.

Vector fields on the doughnut

Your first goal is to construct a pair of vector fields on the doughnut (you can really construct them on all of $\mathbb{R}^3$ if you want, but they only need to make sense on the surface of the doughnut. (Remember, our doughnut has a central radius of 4 meters and an interior radius of 2 meters).

1. Your first vector field $\vec{F}$ should:

(1) Have each vector that starts at the doughnut be tangent to the doughnut.
(2) Point in the direction of that rotates along the larger (radius 4) axis.
(3) The magnitude of each vector is equal to 1 (at least those vectors on the surface).

Your vector field need not be everywhere defined (the most obvious choice is probably not defined along the z-axis).

Your second vector field $\vec{G}$ should:

(1) Have each vector that starts at the doughnut be tangent to the doughnut.
(2) Point in the direction of that rotates along the smaller (radius 2) axis.
(3) The magnitude of each vector is equal to 1 (at least those vectors on the surface).

Your vector field need not be everywhere defined (the most obvious choice is probably not defined along the circle $x^2 + y^2 = 4$ or along the z-axis).

Sketch these vector fields on the surface of the doughnut (perhaps get a computer to sketch the doughnut, then sketch the vector fields on it by hand, or you can get the computer to try to sketch it too which is *tricky*, I can help with this if you want).

2. Explore the following question, are either of these vector fields independent of path of the surface of the doughnut? In particular, choose at least 3 loops C_1, C_2, C_3 on the surface of the doughnut (make sure that some of the loops have different characteristics). Then compute $\int_{C_i} \vec{F} \cdot d\vec{r}$ and $\int_{C_i} \vec{G} \cdot d\vec{r}$.

3. Compute the curl of $\vec{F}$ and $\vec{G}$. Are these zero (at least where they are defined)? (In other words, if $\vec{F}$ and $\vec{G}$ represent differential forms, compute $d\vec{F}$ and $d\vec{G}$).

4. Pullback[1] the differential 1-forms $\vec{F}$ and $\vec{G}$ as well as the differential 2-forms $d\vec{F}$ and $d\vec{G}$ onto the $\alpha\beta$ coordinates of the *surface of the doughnut*. Simplify as much as you (or a computer) can, which if any are zero?

5. Now, find vector fields $\vec{F}_2$ and $\vec{G}_2$ on $\mathbb{R}^3$ that point in the same direction as $\vec{F}$ and $\vec{G}$ (at least on the surface of the doughnut) and so that they pullback to constant vector fields on in $\alpha\beta$ coordinates. Compute integrals of $\vec{F}_2$ and $\vec{G}_2$ along several loops (in particular, compute loops that can be deformed into one another). Do you get the same answer regardless of which loop you consider?

6. The critical mass of sprinkles, at the point $(3\sqrt{2}, 3\sqrt{2}, 0)$, exerts a force, towards the sprinkles, on the ants. The magnitude of the force is the reciprocal of the distance of the ant to the sprinkles to the nth power (for some n). Write down a formula for this vector field. Is it conservative or not (except at the sprinkles)? Does it yield path independent integrals?

Ideas for above and beyond projects.

(1) Perhaps the jelly in the center of the doughnut also creates a force field that acts on the ants. Come up with a reasonable vector field that represents this (it should be larger in magnitude when the ants get near it). It would be more impressive if it had zero curl (where it is defined). How do various path integrals behave with respect to it.
(2) Explore vector fields in the $\alpha\beta$ coordinates, realizing that α and β have the funny property that $0 = 2\pi = 4\pi = \ldots$ and $\pi = 3\pi = \ldots$. Thus functions like $f(\alpha, \beta) = \alpha$ for $\alpha, \beta \in [0, 2\pi)$ are not even continuous. Find some functions that are differentiable and continuous, build some vector fields with them. Can you find interesting vector fields that $d\vec{F} = 0$ but are not conservative?
(3) Explore vector fields on other shapes. For instance, a famous theorem says that any vector field on the surface of the sphere, made up of vectors tangent to the sphere, has some vectors of magnitude 0. While this is easy to believe, it can be tricky to prove. Can you find explicit and interesting tangent vector fields on the sphere? How about one with only 1 place where the tangent vector field has magnitude 0?
(4) Whatever else you can think of.

[1] Pulling back differential forms just means plugging in $x = f(\alpha, \beta)$ and $y = g(\alpha, \beta)$ and then finding differential forms in terms of α and β