F-SINGULARITIES AND FROBENIUS SPLITTING NOTES

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1. Introduction

These notes are from an unofficial topics course taught by Karl Schwede at the University of Utah in the Fall of 2010.

2. Assumptions and notation

Throughout all rings will be Noetherian and excellent. The excellent assumption can in many cases be removed, but for simplicity we will keep it.

Often rings will be assumed to contain a field of characteristic $p > 0$. If R is a ring of characteristic $p > 0$, it possesses that absolute Frobenius map $F : R \to R$. This is the map defined by $F(r) = r^p$ it is a map of rings. It thus turns R into an R-module with a non-standard action. That is, $r.x = r^p x$. We denote this R-module by $F_* R$. Why? Well, if $X = \operatorname{Spec} R$, then $F : \mathcal{O}_X \to F_* \mathcal{O}_X$ is the structural map associated to Frobenius. There are other common notations as well.

(a) ${}^1 R$.
(b) $R^{1/p}$ if R is reduced.

You may notice the number 1 in front of the R, and wonder why it's there. The point is that you can iterate Frobenius $F^e = F \circ F \circ \cdots \circ F$ and have induced module structures on R, denoted by $F^e_* R \cong {}^e R \cong R^{1/p^e}$. It is useful to observe that F^e_* is an *exact functor*.

These different notations for the same thing have different advantages. $R^{1/p}$ is useful because it allows one to easily distinguish elements from R and $F_* R$. On the other hand, it can lead to confusing statements since if we view $I^{1/p} \subseteq R^{1/p}$ as the ideal of $R^{1/p}$ made up of pth roots of I, then $(I^{1/p})^p = (I^p)^{1/p} \neq I$ (the latter is an ideal of R, where the two former are ideals of $R^{1/p}$). $I^{1/p}$ also is not a decent notation for modules.

Definition 2.1. Given an ideal $(x_1, \dots, x_n) = I \subseteq R$, we use $I^{[p^e]}$ to denote the ideal $(x_1^{p^e}, \dots, x_n^{p^e})$.

It is easy to see that this definition is independent of the choice of generators of I since $I^{[p^e]}$ can also be identified with the $F^e_* R$-ideal $I \cdot (F^e_* R)$.

Example 2.2. Consider the ring $R = \mathbb{F}_p[x_1, \dots, x_n]$. Then $F_* R$ is a free R-module with basis $\{x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} | 0 \leq \lambda_i \leq p - 1\}$.

The object $F_* R$ plays well with localization and completion.

Lemma 2.3. *Suppose that R is a ring of characteristic $p > 0$, $\mathfrak{m}$ is a maximal ideal and W is a multiplicative set. Then*

(i) $W^{-1}(F_*R) \cong F_*(W^{-1}R)$

(ii) $F_*\hat{R} \cong \widehat{F_*R}$ *(where the second is completion as an R-module).*

where the $\hat{\_}$ denotes completion with respect to $\mathfrak{m}$.

Proof. The first statement follows since $W^{-1}(F_*R) = F_*((W^p)^{-1}R)$ but $(W^p)^{-1}R \cong W^{-1}R$ since $r/w = (rw^{p-1})/w^p$. For (ii), notice first that $\hat{R} = \lim_{\leftarrow} R/\mathfrak{m}^n = \lim_{\leftarrow} R/(\mathfrak{m}^n)^{[p^e]}$ since the two sequences of ideals are cofinal. Then

$$\widehat{F^e_*R} = \lim_{\leftarrow}(F^e_*R)/\mathfrak{m}^n = \lim_{\leftarrow} F^e_*(R/(\mathfrak{m}^n)^{[p^e]}) = F^e_* \lim_{\leftarrow}(R/(\mathfrak{m}^n)^{[p^e]}) = F^e_* \lim_{\leftarrow} R/\mathfrak{m}^n = F^e_*\hat{R}.$$

□

Of course, there is another functor also, F^* which is defined by $F^*\mathscr{L} = \mathscr{L} \otimes F_*\mathcal{O}_X$ (and then viewed as an $F_*\mathcal{O}_X = \mathcal{O}_X$ module via the action on the right). Unlike F_*, F^* is not exact in general (although it sometimes is, as we will see). If $\mathscr{L}$ is a line bundle, then $F^*\mathscr{L} = \mathscr{L}^p$. One can see this by looking at the transition functions and noticing that they are raised to powers.

Definition 2.4. A ring of characteristic $p > 0$ is said to be *F-finite* if the Frobenius map is a finite map. In other words, if R is reduced, this means that $R^{1/p}$ is a finite R-module.

Lemma 2.5. *If R is F-finite, so is any quotient, localization, or completion at a maximal ideal.*

Proof. Suppose that R is F-finite, thus we have a surjective map of R-modules $\oplus_{i=1}^n R \to F_*R$ for some n. If W is a multiplicative set then tensoring with $W^{-1}R$ will give us a new surjection. Completion is similar and quotienting out by an ideal is also straightforward. □

Note that thus if you start with a variety over an algebraically closed (or perfect) field, anything you might ever end up working with is still F-finite (even if you eventually move beyond having perfect residue fields) because $k[x_1, \dots, x_n]$ is F-finite (as long as k is an F-finite, eg perfect, field). The usual examples of non-perfect fields, $\mathbb{F}_p(x)$ are still F-finite! Although $\mathbb{F}_p(x_1, \dots, x_n, \dots)$ is not F-finite.

Technical lemmas we won't prove.

Lemma 2.6. [Kun76][Gab04] *If R is F-finite then R is excellent and it has a dualizing complex.*

Remark 2.7. If you don't know what a dualizing complex is, don't worry about it.

In other words, if you assume F-finite, you're working in a pretty geometric setting already.

3. Flatness of Frobenius

Suppose that R is a noetherian ring of characteristic $p > 0$. In [Kun69b], Kunz noticed the following: If $F_* R$ is flat as an R-module and $R \subseteq S$ is unramified in codimension 1, then $R \subseteq S$ is unramified.

Definition 3.1. An extension $R \subseteq S$ is called *unramified* is for every $\mathfrak{q} \in S$ with $\mathfrak{p} = \mathfrak{q} \cap R$, one has that $\mathfrak{p}S = \mathfrak{q}S$ and also that $k(\mathfrak{p}) \subseteq k(\mathfrak{q})$ is separable.

He then noticed that the condition that $F_* R$ is a flat R-module is equivalent to R being regular.

Theorem 3.2. [Kun69a] *Suppose that R is a local ring of characteristic $p > 0$. Then R is regular if and only if $F_* R$ is flat as an R-module.*

Proof. [Kun69a] We'll only prove the ($\Rightarrow$) direction today. We do not assume that R is F-finite. Suppose that R is regular, then $\hat{R}$ is a power series ring $k[[x_1, \ldots, x_n]]$ where k is the residue field of R. We have the following diagram:

$$
\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[dd, hook, "\alpha"'] & \hat{R} \arrow[r, equal] & k[[x_1, \ldots, x_n]] \arrow[d, hook] \\
 & & k^{1/p}[[x_1, \ldots, x_n]] \arrow[d, hook] \\
R^{1/p} \arrow[r, "\gamma"] & \hat{R}^{1/p} \arrow[r, equal] & k^{1/p}[[x_1^{1/p}, \ldots, x_n^{1/p}]]
\end{tikzcd}
$$

Once we show that the right vertical column is flat, then we know that $\gamma \circ \alpha$ is also flat. This combined with the fact that γ is faithfully flat implies that α is flat by [Mat89, Page 46].

So, we need to show that the right vertical column is flat. The inclusion $k^{1/p}[[x_1, \ldots, x_n]] \subseteq k^{1/p}[[x_1^{1/p}, \ldots, x_n^{1/p}]]$ is clearly flat since the target is free as an R-module. The other inclusion is also free since $k^{1/p}$ is a flat k-module (this requires a little bit of work, Kunz cites [Bou98, Chapter III, Section 5]). $\square$

Thus on a regular variety X, $F^e_* \mathcal{O}_X$ is a locally free sheaf (of F-finite rank assuming that X is F-finite). In particular, F^* is an exact functor if and only if X is regular.

Proposition 3.3. *If $X = \operatorname{Spec} R$ is an F-finite regular affine scheme, then $F^e : \mathcal{O}_X \to F^e_* \mathcal{O}_X$ splits as a map of $\mathcal{O}_X$-modules.*

Proof. First we claim that the statement is local. Indeed, consider the map $\sigma : \operatorname{Hom}_R(F^e_* R, R) \to R$ defined by evaluation at 1. The map F^e defined in the statement of the proposition splits if and only if σ surjects. The surjectivity of σ is a local property (since R is F-finite), so we can assume that R is local. Thus $F^e_* R$ is a flat and thus free R-module. Therefore there exist many surjective maps $\phi : F^e_* R \to R$ (project onto one component) we just need to see that one of them is a splitting. Suppose $\phi(x) = 1$ for some $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ and some $x \in F^e_* R$, but then $\psi(\_\_) = \phi(x \cdot \_\_)$ clearly is a splitting of F^e. $\square$

The splitting of Frobenius is a statement about the singularities of X. If it occurs, it says something about the singularities being mild (we'll see some very effective criteria for checking this in a couple days).

Example 3.4. Let us compute $F^e_* \mathcal{O}_X$ on $X = \mathbb{P}^1_k$, where $k = \bar{k}$. We know that $F^e_* \mathcal{O}_X = \mathcal{O}_X(a_1) \oplus \mathcal{O}_X(a_2) \oplus \cdots \oplus \mathcal{O}_X(a_{p^e})$ because we are working on $\mathbb{P}^1$. We also know that $H^0(X, F^e_* \mathcal{O}_X) = k$ so exactly one of the a_i is equal to zero (and the rest are negative), say $a_1 = 0$. We will show that the rest of the $a_i = -1$, to see this consider

$$k^{p^e+1} = H^0(X, (F^e_* \mathcal{O}_X(p^e))) = H^0(X, (F^e_* \mathcal{O}_X) \otimes \mathcal{O}_X(1))$$
$$= H^0(X, \mathcal{O}_X(a_1 + 1) \oplus \mathcal{O}_X(a_2 + 1) \oplus \cdots \oplus \mathcal{O}_X(a_{p^e} + 1)) \geq k^{2+(a_2+2)+\cdots+(a_{p^e}+2)}.$$

But the only way this will happen is if each $a_i = -1$ for $i \geq 2$ (since they all already negative numbers).

For $X = \mathbb{P}^1$, we saw that $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ is also going to split (because 1 goes to 1). However, not all smooth varieties which have locally split Frobenius have globally split Frobenius. Projective space does (as we'll see, as do toric varieties in general and Fano varieties in "most" characteristics).

Example 3.5. Suppose that X is a supersingular elliptic curve, see [Har77, Chapter IV, Section 4, page 332], in other words $F : H^1(X, \mathcal{O}_X) \to H^1(X, F_* \mathcal{O}_X)$ is the zero map. Then X is not Frobenius split. To prove it, observe that $H^1(X, \_\_)$ is a functor. On the other hand, one can show that if X is an ordinary elliptic curve, it is Frobenius split (more on this later).

Frobenius split varieties satisfy strong properties.

Lemma 3.6. *Suppose that X is a variety whose Frobenius morphism splits. Then for any ample line bundle $\mathscr{L}$ on X, $H^i(X, \mathscr{L}) = 0$ for all $i \geq 0$.*

Proof. Note that $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ splitting implies that $\mathscr{L} \otimes \mathcal{O}_X \to \mathscr{L} \otimes F^e_* \mathcal{O}_X = F^e_*(\mathcal{O}_X \otimes (F^e)^* \mathscr{L}) = F^e_* \mathscr{L}^{p^e}$ also splits. We then have $H^i(X, \mathscr{L}) \to H^i(X, F^e_* \mathscr{L}^{p^e})$ injects. But the right side vanishes by Serre vanishing for $e \gg 0$ so thus the left side vanishes too. $\square$

Even though Kodaira vanishing fails in positive characteristic, it holds for Frobenius split varieties.

Theorem 3.7. *Suppose that X is a projective Frobenius split variety. Then for any ample line bundle $\mathscr{L}$ on X, $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for $i > 0$.*

Proof. It's not hard, but we'll prove it a little later. $\square$

We also briefly mention a link to projective normality.

Definition 3.8. Suppose that $Y \subseteq X$ is a closed subvariety of X. Given a map $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$, we say that Y is *ϕ-compatible* if ϕ induces a map $\bar{\phi} : F^e_* \mathcal{O}_Y \to \mathcal{O}_Y$ by restriction.

Theorem 3.9. *If $\phi : \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}$ is a splitting of Frobenius, then any ϕ-compatible normal $Y \subseteq \mathbb{P}^n$ is embedded in $\mathbb{P}^n$ projectively normally.*

Proof. It is sufficient to show that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(i)) \to H^0(Y, \mathcal{O}_Y(i))$ is surjective for all i (see [Har77, Chapter II, Exercise 5.14]). Consider the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p^e i)) & \xrightarrow{\phi(i)} & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(i)) \\ \gamma \downarrow & & \downarrow \delta \\ H^0(\mathbb{P}^n, \mathcal{O}_Y(p^e i)) & \xrightarrow{\bar{\phi}(i)} & H^0(\mathbb{P}^n, \mathcal{O}_Y(i)) \end{array}$$

By Serre vanishing, γ is surjective and $\bar{\phi}(i)$ is also surjective because it is induced from a splitting. Thus δ is surjective as well. ☐

4. Flatness of Frobenius implies regular

Today, we'll complete the proof that having a flat Frobenius map implies that X is regular (a result of Kunz).

Theorem 4.1. *Suppose that X is a scheme, then R is regular if and only if $F^e_* \mathcal{O}_X$ is flat as an $\mathcal{O}_X$-module for some $e > 0$.*

Proof. We'll need several lemmas, but let us sketch the proof first. The statement is local so we may assume that $X = \operatorname{Spec} R$ where $(R, \mathfrak{m})$ is a local ring. Write $\mathfrak{m} = (x_1, \ldots, x_n)$ where the x_i are a minimal system of generators. Our goal is to show that $n = \dim R$.

First observe that it is harmless to replace e by ne for any integer $n > 0$. Unlike what I said in class, the proof works fine for non-algebraically closed residue fields.

Step 1. $\mathfrak{m}^{[p^e]}/(\mathfrak{m}^{[p^e]})^2$ is a free R-module.
Step 2. Apply lemmas of Lech to conclude that $l_R(R/\mathfrak{m}^{[p^e]}) = p^{ne}$ for all $p \in N$.
Step 3. Assume R is complete and write $R = S/\mathfrak{a} = k[[x_1, \ldots, x_n]]/\mathfrak{a}$. Then notice that $l_S(S/\mathfrak{m}_S^{[p^e]}) = p^{ne}$ for all $e \geq 0$. But this implies that $\mathfrak{a} = 0$ and so $R = S$. This actually completes the proof of step 3.

We begin with the proof of step 1.

$$F_* \mathfrak{m}^{[p^e]}/(\mathfrak{m}^{[p^e]})^2 = (\mathfrak{m}/\mathfrak{m}^2) \otimes_R F_* R = (\mathfrak{m}/\mathfrak{m}^2) \otimes_(R/\mathfrak{m}) F_*(R/\mathfrak{m}^{[p^e]})$$

because of flatness of $F_* R$ over R. But the right side is a free $F_*(R/\mathfrak{m}^{[p^e]})$-module. This implies that the (minimal set of) generators $x_1^{p^e}, \ldots, x_n^{p^e}$ of $\mathfrak{m}^{[p^e]}$ are *Lech-independent.*

Definition 4.2. That a sequence of elements $f_1, \dots, f_n \in R$ is called *Lech-independent* if for any $a_1, \dots, a_n \in R$ such that $a_1 x_1^{p^e} + \dots + a_n x_n^{p^e} = 0$, then $a_i \in \mathfrak{m}^{[p^e]}$.

We now begin step 2. For this, we begin with a Lemma.

Lemma 4.3. [Lec64, Lemma 3] *If $f_1, \dots, f_n$ are Lech-independent elements and $f_1 \in gR$ for some $g \in R$, then $g, f_2, \dots, f_n$ is also Lech-independent. Furthermore, $(f_2, \dots, f_n) : g \subseteq (f_1, \dots, f_n)$*

Proof. Write $f_1 = gh$. Suppose $a_1 g + \dots + a_n f_n = 0$ multiplying the equation through by h implies that $a_1 \in (f_1, \dots, f_n) \subseteq (g, \dots, f_n)$ (this also proves the second statement of the theorem). Say $a_1 = b_1 f_1 + \dots + b_n f_n$. Plugging this in, we get that

$$0 = (b_1 f_1 + \dots + b_n f_n) g + a_2 f_2 + \dots + a_n f_n = b_1 g f_1 + (b_2 g + a_2) f_2 + \dots + (b_n g + a_n) f_n.$$

Therefore, $b_i g + a_i \in (f_1, \dots, f_n) \subseteq (g, f_2, \dots, f_n)$ for $i \geq 2$ and so $a_i \in (g, f_2, \dots, f_n)$ for $i \geq 2$ as desired. $\square$

This lemma, combined with the fact that $x_1^{p^e}, \dots, x_n^{p^e}$ are Lech-independent, proves that $x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ are Lech-independent for $\alpha_i \leq p^e$ (or basically for any α_i since we can make e bigger). We now need another Lemma.

Lemma 4.4. [Lec64, Lemma 4] *If $f_1, \dots, f_n$ are Lech-independent and $f_1 = gh$. Then*

$$l_R\left(R/(f_1, \dots, f_n)\right) = l_R\left(R/(g, f_2, \dots, f_n)\right) + l_R\left(R/(h, f_2, \dots, f_n)\right).$$

Proof. First notice that

$$l_R\left(R/(f_1, \dots, f_n)\right) = l_R\left(R/(g, f_2, \dots, f_n)\right) + l_R\left((g, f_2, \dots, f_n)/(f_1, \dots, f_n)\right)$$

.

However,

$$(g, f_2, \dots, f_n)/(f_1, \dots, f_n) = (gR + (f_1, \dots, f_n))/(f_1, \dots, f_n) \cong R/((f_1, \dots, f_n) : gR)$$

. We certainly know that $(f_1, \dots, f_n) : gR \supseteq (h, f_2, \dots, f_n)$ and we will show the converse inclusion. Suppose then that $ag = a_1 f_1 + \dots + a_n f_n$, then $(a_1 h - a) g + a_2 f_2 + \dots + a_n f_n = 0$, so that the $a_1 h - a \in (f_2, \dots, f_n) : g \subseteq (f_1, \dots, f_n)$. But then $a_1 h - a = b_1 f_1 + \dots + b_n f_n = b_1 g h + \dots + b_n f_n$ which implies that $a \in (h, b_2, \dots, b_n)$. $\square$

We will explain how this lemma implies (inductively) that $l_R(R/\mathfrak{m}^{[p^e]}) = p^{ne}$ as desired. We will show that $l_R\left(R/(x_1^{\alpha_1}, \dots, x_n^{\alpha_n})\right) = \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ by induction on $\sum_i \alpha_i$. The base case is obvious.

If $\alpha_i > 1$, by the previous lemma, we know that

$$\begin{aligned} &l_R\left(R/(x_1^{\alpha_1}, \dots, x_n^{\alpha_n})\right) \\ = l_R\left(R/(x_1^{\alpha_1}, \dots, x_{i-1}^{\alpha_{i-1}}, x_i^1, x_{i+1}^{\alpha_{i+1}}, \dots, x_n^{\alpha_n}\right) + l_R\left(R/(x_1^{\alpha_1}, \dots, x_{i-1}^{\alpha_{i-1}}, x_i^{\alpha_i - 1}, x_{i+1}^{\alpha_{i+1}}, \dots, x_n^{\alpha_n})\right) \\ = (\alpha_1 \cdot \dots \cdot \alpha_{i-1} \cdot 1 \cdot \alpha_{i+1} \cdot \dots \cdot \alpha_n) + (\alpha_1 \cdot \dots \cdot \alpha_{i-1} \cdot (\alpha_i - 1) \cdot \alpha_{i+1} \cdot \dots \cdot \alpha_n) \\ = \alpha_1 \cdot \dots \cdot \alpha_n \end{aligned}$$

which completes the induction.

Finally, we do step 3 (which we already did). $\square$

5. Criteria for local Frobenius splitting I (Fedder's criteria)

Today, we'll learn about a result called for the second statement, assume that $ag+a_2f_2+\dots+a_nf_n = 0$, so Fedder's criteria for local Frobenius splitting. We'll also explore Frobenius splitting of projective varieties vs Frobenius splitting of graded rings.

First local behavior. Suppose that S is an F-finite regular ring such that F_*S is a free S-module (for example, this happens if S is local). Write $R = S/I$. Suppose that $\phi : F^e_*R \to R$ is R-linear. Consider the following diagram where the vertical arrows are the natural quotients:

$$\begin{tikzcd} F^e_*S \arrow[r, dotted, "\psi"] \arrow[d] & S \arrow[d] \\ F^e_*R \arrow[r, "\phi"'] & R \end{tikzcd}$$

Because F^e_*S is free and thus projective, there exists a F^e_*S-module map ψ as labeled in the diagram (which makes the diagram commute). This map is not unique! If we further assume that S is local, then if ϕ is surjective, then so must be ψ (since if $\psi(S) \subseteq \mathfrak{m}_S$, then $\phi(S/I) \subseteq \mathfrak{m}_S/I = \mathfrak{m}_R \subsetneq R$.

Lemma 5.1. *With the notation as above, if R has a Frobenius splitting $\phi : F^e_*R \to R$ (ie, an R-linear map that sends 1 to 1), then there is a Frobenius splitting ψ' on S which also induces a (possibly different) Frobenius splitting on R as in the diagram above.*

Proof. We already saw the existence of a map $\psi : F^e_*S \to S$ which is surjective. Suppose that $\psi(x) = 1$. Then consider the map $\psi : F^e_*S \to S$ defined by the rule $\psi'(\_\_) = \psi(x \cdot \_\_)$, this is clearly a splitting. This map still induces a map on R (defined by $\phi'(\_\_) = \phi(\bar{x} \cdot \_\_)$) and it is a splitting since ψ' is). $\square$

This suggests that in order to study the (possible) existence of F-splittings of R it might be good to study the splittings on S which induce splittings on R. First suppose that S is a regular local ring, let us study the maps $\phi \in \mathrm{Hom}_S(F^e_*S, S)$. To do this, I'd like to describe a little bit of duality for a finite map (Frobenius being the finite map).

In order to do this, we need a little bit of theory. So let's quickly review (Grothendieck) duality for a finite map.

Definition 5.2. Suppose that R is a local ring with a normalized dualizing complex $\omega_R^{\bullet}$. Then the *canonical module* ω_R of R is $\mathcal{H}^{-\dim R}(\omega_R^{\bullet})$. A canonical module on an arbitrary ring/scheme is a module whose localization is isomorphic the canonical module at every prime/point.

Somewhat more explicitly, we can define the canonical module of R as follows. If X is a normal irreducible scheme of (essentially) finite type over a field. One can define ω_X as follows:

$$\omega_X = \left(\wedge^{\dim X}\Omega^1_{X/k}\right)^{**}.$$

Here the symbol ** means apply the functor $\mathrm{Hom}_R(\_\_, R)$ twice.

Definition 5.3. A divisor K_X on a normal scheme X such that $\mathcal{O}_X(K_X) \cong \omega_X$ is called a *canonical divisor*.

Canonical divisors are divisor classes on varieties over fields. This is much more ambiguous on general schemes since ω_X can be twisted by any line bundle and still be a canonical module (we only defined it locally).

Theorem 5.4. [Har66] *Let $R \subseteq S$ be a finite inclusion of rings with dualizing complexes and that ω_R is a canonical module for R. Then:*

(i) $\mathrm{Hom}_R(S, \omega_R)$ *is a canonical module for S and if we are working with varieties of finite type over a field, we may assume that the canonical module constructed in this way for S, agrees with the one obtained by taking wedge-powers of $\Omega_{X/k}$.*
(ii) *If N is an S-module, then we have an isomorphism of S-modules* $\mathrm{Hom}_R(N, \omega_R) \cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, \omega_R)) \cong \mathrm{Hom}_S(N, \omega_S)$.

Remark 5.5. The functor $\mathrm{Hom}_R(S, \_\_)$ is often called $f^{\flat}$ or $f^!$ where $f : \mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is the induced map.

We will apply this theorem to the case of the Frobenius map.

Corollary 5.6. *Suppose that X is a normal scheme of essentially finite type over an F-finite field (or $X = \mathrm{Spec}\, R$ where R is an F-finite normal local ring). Then $\mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X((1-p^e)K_X)$.*

Proof. Let U denote the regular locus of X so that $X \setminus U$ is codimension 2 or higher. By basic facts about the reflexive sheaves, see for example [Har94], it is enough to show this isomorphism with X replaced by U (in other words, we may assume that X is regular). We may write

$$\begin{aligned} &\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \\ \cong\ &\mathscr{H}\mathrm{om}_{\mathcal{O}_X}((F^e_* \mathcal{O}_X) \otimes \mathcal{O}_X(K_X), \mathcal{O}_X(K_X)) \\ \cong\ &\mathscr{H}\mathrm{om}_{\mathcal{O}_X}((F^e_* \mathcal{O}_X(p^e K_X)), \mathcal{O}_X(K_X)) \\ \cong\ &\mathscr{H}\mathrm{om}_{F^e_* \mathcal{O}_X}(F^e_* \mathcal{O}_X(p^e K_X), F^e_* \mathcal{O}_X(K_X)) \\ \cong\ &F^e_* \mathcal{O}_X((1-p^e)K_X). \end{aligned}$$

The funny hypotheses at the start of this proof are there to insure that $s\,\mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X(K_X))$ is isomorphic to $\mathcal{O}_X(K_X)$ (and not some other canonical module). $\square$

This greatly restricts which varieties can be globally Frobenius split.

Corollary 5.7. *Suppose that X is a Frobenius split variety, then $H^0(X, \mathcal{O}_X(-nK_X)) \neq 0$ for some $n > 0$. In particular, X cannot be projective and of general type.*

Proof. If X is Frobenius split then $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X((1-p^e)K_X)$ is non-zero for some ϕ. In fact, one can take $e = 1$ and so $n = p - 1$. $\square$

Another interesting conclusion of this is the following.

Corollary 5.8. *Suppose that $X = \mathrm{Spec}\, R$ where R is a normal F-finite local ring. If $\mathcal{O}_X((1-p^e)K_X)$ is locally free, then $\mathcal{O}_X((1-p^e)K_X)$ is also locally free and thus isomorphic to $\mathcal{O}_X$ (this happens for example if R is Gorenstein). In particular, $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ is a cyclic $F^e_* \mathcal{O}_X$-module. A $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ which generates $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ is called a* generating homomorphism*.*

Example 5.9. If $X = \mathrm{Spec}\, k[x_1, \dots, x_n]$, then the map which sends $(x_1 \dots x_n)^{p^e-1}$ to 1 and the other relevant monomials to zero, is a "generating map". In the local case, there are other generating maps as well (send some of the other monomials to non-zero things).

Now we need some notation.

Definition 5.10. Suppose that S is a ring and I is an ideal. If $\psi : F^e_* S \to S$ is an S-linear map, we say that I is *ϕ-compatible* if $\psi(F^e_* I) \subseteq I$.

Remark 5.11*.* Clearly if I is ψ-compatible, then ψ induces a map on R/I.

Remark 5.12*.* Remember that for ideals I, J, the notation $I : J$ is all the elements $r \in R$ such that $rJ \subseteq I$. In other words, it is the same as $Ann_R(J + I/I)$.

Theorem 5.13. [Fed83]*[Fedder's Lemma] Suppose that S is a regular local ring and that $R = S/I$. The set of $\phi \in \mathrm{Hom}_S(F^e_* S, S)$ which satisfy $\phi(F^e_* I) \subseteq I$ is equal to $F^e_*(I^{[p^e]} : I) \cdot \mathrm{Hom}_S(F^e_* S, S) \cong F^e_*(I^{[p^e]} : I)$ and those which induce the zero map on $R = S/I$ correspond to $I^{[p^e]}$. In conclusion, $\mathrm{Hom}_R(F^e_* R, R) \cong F^e_*(I^{[p^e]} : I)/(I^{[p^e]})$.*

Proof. Let $\Phi \in \mathrm{Hom}_S(F^e_* S, S)$ be a generating map. We will first show the following lemma.

Lemma 5.14. *For any ideals $I, J \subseteq S$, we have $\Phi(F^e_* J) \subseteq I$ if and only if $I^{[p^e]} \supseteq J$.*

Proof. The ($\Leftarrow$) direction is easier and we start with that. We claim that $\phi(F^e_* I^{[p^e]}) \subseteq I$. To see this, note that if $I = (x_1, \ldots, x_n)$, then $I^{[p^e]} = (x_1^{p^e}, \ldots, x_n^{p^e})$ and so if $z \in I^{[p^e]}$, then $z = \sum a_i x_i^{p^e}$. Then $\Phi(z) = \Phi(\sum a_i x_i^{p^e}) = \sum x_i \phi(a_i)$. The first direction then immediately follows.

Conversely, suppose that $\Phi(F^e_* I) \subseteq J$. We choose $y_1, \ldots, y_m$ to be a basis for $F^e_* S$ over S (we can obviously project on to each factor via multiplication of Φ by elements of $F^e_* S$, and any map $\phi : F^e_* S \to S$ is a sum of such projections). So, we need $F^e_* I \subseteq \oplus J \cdot y_i = J \cdot F^e_* S = F^e_* J^{[p^e]}$. In other words, $I \subseteq J^{[p^e]}$ as desired. $\square$

I claim that a map $\phi : F^e_* S \to S$ sends $F^e_* I$ into I if and only if $\phi \in F^e_*(I^{[p^e]} : I) \cdot \Phi$. To see this, write $\phi = z \cdot \Phi$ for some $z \in F^e_* S = S$. Then $\phi(F^e_* I) \subseteq I$ if and only if $\Phi(F^e_* z I) \subseteq I$ which happens if and only if $zI \subseteq I^{[p^e]}$, in other words, if and only if $z \in I^{[p^e} : I$. Thus $\phi \in F^e_*(I^{[p^e]} : I) \cdot \Phi$ if and only if $\phi(F^e_* I) \subseteq I$.

For the second statement, suppose that $\phi \in I^{[p^e]} \cdot \Phi$. Thus for every $x \in F^e_* S$, $\phi(x) \in I$ (use the previous lemma with $J = I^{[p^e]}$ $I = I$). Thus the induced map on $R = S/I$ is the zero map. Conversely, suppose that $\phi \in F^e_*(I^{[p^e]} : I) \cdot \Phi$ but $\phi \notin I^{[p^e]} \cdot \Phi$. Thus there is some $x \in F^e_* S$ such that $\phi(x) \notin I$ and so the induced map on $R = S/I$ is non-zero. $\square$

Corollary 5.15 (Fedder's criteria)**.** *If $(S, \mathfrak{m})$ is a F-finite regular local ring and $R = S/I$, then R is F-split if and only if $I^{[p^e]} : I$ is not contained in $\mathfrak{m}^{[p^e]}$.*

Proof. For $\bar{\phi} \in \mathrm{Hom}_R(F^e_* R, R)$ (induced from $\phi : F^e_* S \to S$) to be surjective, it must contain 1 in it's image. This happens if and only if $\phi \notin \mathfrak{m}^{[p^e]} \cdot \Phi$ (where Φ is in the previous proof). Such a map exists if and only if $I^{[p^e]} : I \not\subseteq \mathfrak{m}^{[p^e]}$. $\square$

Remark 5.16. If $I = (f)$ is a principal ideal, then $I^{[p^e]} : I = (f^{p^e - 1})$ which is very easy to compute by hand. In many cases, the colon's can be done via a computer.

We now do several examples.

Example 5.17. The following rings are F-split.

1. $R = k[x_1, \ldots, x_n]/(x_1 \cdot \cdots \cdot x_n)$. Notice that $(x_1 \cdot \cdots \cdot x_n)^{p^e - 1} \not\subseteq (x_1^{p^e}, \ldots, x_n^{p^e}) = m^{[p^e]}$.
2. $R = k[x, y, z]/(x^2 - yz)$. Notice that $(x^2 - yz)^{p^e - 1}$ has a term $(yz)^{p^e - 1}$ which does not appear in $m^{[p^e]}$.
3. $R = k[x, y, z]/(x^2 - y^2 z)$ if the characteristic of k is not 2. In this case, $(x^2 - y^2 z)^{p-1}$ has a term $\binom{p-1}{(p-1)/2} x^{p^e - 1} y^{p^e - 1} z^{\frac{p^e - 1}{2}}$ and so the question is

whether p divides the binomial coefficient. But it is clear that it does not.

(4) $R = k[x, y, z]/(x^3 + y^3 + z^3)$ if the characteristic of k is 7 (check it yourself). One can also check that it is not F-split for characteristics $2, 3, 5$ and more generally if $p = 2 \mod 3$.

Fedder's Lemma suggests the following question.

Question 5.18. Given an arbitrary ring T with quotient $R = T/I$. Is it true that every map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is induced from a map $\phi \in \mathrm{Hom}_T(F^e_* T, T)$?

The answer to this question is no as the following example demonstrates:

Example 5.19. Consider $S = k[x, y, z]$, $T = k[x, y, z]/(x^2 - yz)$ and $R = k[x, y, z]/(x, y)$. The map $\Phi_R : F_* R \to R$ which sends z^{p-1} to 1 and the other z^i to zero is induced by maps written as $\Phi_S(w \cdot \_\_)$ where Φ_S is the $F_* S$-module generator of $\mathrm{Hom}_S(F_* S, S)$ discussed above and w is an element of the coset $(xy)^{p-1} + (x^p, y^p)$. We have to ask ourselves whether any such w can be inside $((x^2 - yz)^{p-1}) + (x^p, y^p)$, and the answer is clearly no.

6. Very basic facts about Frobenius splitting

First we discuss the difference between F-purity and F-splitting.

Definition 6.1. A ring R of characteristic $p > 0$ is said to be *F-pure* if for every R-module M, the map $M \otimes R \to M \otimes F_* R$ is pure.

Clearly an F-split ring is F-pure. Furthermore, if R is F-finite, then an F-pure ring is also F-split (see The notion of F-purity is much better behaved outside the F-finite context. However, we won't be going there.

In an F-finite scheme, F-purity is used interchangeably with local F-splitting. An F-splitting (without a "local" qualifier) is always viewed as a global statement.

Here we list (and prove) a number of basic facts about Frobenius splittings, again mostly in the local context.

Theorem 6.2. *Suppose that R is an F-finite ring. Then the following hold:*

(a) *If R is Frobenius split (F-split) then R is reduced.*
(b) *If R_Q is Frobenius split for some $Q \in \operatorname{Spec} R$, then R is Frobenius split in a neighborhood of Q.*
(c) *R is F-split if and only if $R_{\mathfrak{m}}$ is F-split for every maximal ideal $\mathfrak{m}$ if and only if R_Q is F-split for every prime ideal Q.*
(d) *If $R \subseteq S$ is a split inclusion of rings and S is F-split, then R is also F-split.*
(e) *If R is F-split, then for every minimal prime $\mathfrak{q} \subseteq R$, $R/\mathfrak{q}$ is also F-split.*
(f) *If $\phi : F^e_* R \to R$ is any R-linear map and I and J are ϕ-compatible ideals, then so is $I + J$, $I \cap J$, $\sqrt{I}$, and also $I : \mathfrak{a}$ for any ideal $\mathfrak{a}$.*

7. (Weak/Semi)normality and Frobenius splitting

Today we'll prove that a F-split ring is weakly normal and thus seminormal (so first I'll define these terms).

First we'll talk about some hand-wavy geometry. Seminormality (and weak normality) are ways of forcing all gluing of your scheme is as transverse as possible. So first what is "gluing"?

Suppose that R is an F-finite reduced ring with normalization R^N (domain of finite type over a field is fine). The semi-normalization R^{SN} (and weak normalization R^{WN} of R is a partial normalization of R inside R^N). Since R is F-finite it is excellent, and so all these extensions are finite extensions (ie, we don't have to worry about extreme funny-ness).

Definition 7.1. [AB69], [GT80], [Swa80] A finite integral extension of reduced rings $i : A \subset B$ is said to be *subintegral* (respectively *weakly subintegral*) if

(i) it induces a bijection on the prime spectra, and
(ii) for every prime $P \in \operatorname{Spec} B$, the induced map on the residue fields, $k(i^{-1}(P)) \to k(P)$, is an isomorphism (respectively, is a purely inseparable extension of fields).

Remark 7.2. A subintegral extension of rings has also been called a quasi-isomorphism; see for example [GT80].

Remark 7.3. Condition (ii) is unnecessary in the case of extensions of rings of finite type over an algebraically closed field of characteristic zero.

Definition 7.4. [GT80, 1.2], [Swa80, 2.2] Let $A \subset B$ be a finite extension of reduced rings. Define ${}^+_BA$ to be the (unique) largest subextension of A in B such that $A \subset {}^+_BA$ is subintegral. This is called the *seminormalization of A inside B*. A is said to be *seminormal in B* if $A = {}^+_BA$. If A is seminormal inside its normalization, then A is called *seminormal.*

Definition 7.5. [AB69], [Yan85], [RRS96, 1.1] Let $A \subset B$ be a finite extension of reduced rings. Define ${}^*_BA$ to be the (unique) largest subextension of A in B such that $A \subset {}^*_BA$ is weakly subintegral. This is called the *weak normalization of A inside B*. A is said to be *weakly normal in B* if $A = {}^*_BA$. If A is weakly normal inside its normalization, then A is called *weakly normal.*

Remark 7.6. Note the following set of implications.

$$\text{Normal} \Longrightarrow \text{Weakly Normal} \Longrightarrow \text{Seminormal}$$

Consider the following examples:

(i) The union of two axes in $\mathbb{A}^2$, $\operatorname{Spec} k[x,y]/(xy)$, is both weakly normal and seminormal, but not normal (an irreducible node is seminormal as well).
(ii) The union of three lines through the origin in $\mathbb{A}^2$, $\operatorname{Spec} k[x,y]/(xy(x-y))$, is neither seminormal nor weakly normal.

(iii) The union of three axes in $\mathbb{A}^3$, $\operatorname{Spec} k[x,y,z]/(x,y)\cap(y,z)\cap(x,z)$, is both seminormal and weakly normal. In fact, it is isomorphic to the seminormalization of (ii).

(iv) The pinch point $\operatorname{Spec} k[a,b,c]/(a^2b-c^2) \cong \operatorname{Spec} k[x^2,y,xy]$ is both seminormal and weakly normal as long as the characteristic of k is not equal to two. In the case that $\operatorname{char} k = 2$, then the pinch point is seminormal but not weakly normal. Notice that if $\operatorname{char} k = 2$ then the inclusion $k[x^2,y,xy] \subset k[x,y]$ induces a bijection on spectra. Furthermore the induced maps on residue fields are isomorphisms at all closed points. However, at the generic point of the singular locus $P = (y, xy)$, the induced extension of residue fields is purely inseparable. This proves that it is not weakly normal.

(v) $\mathbb{R}[x,y]/(x^2+y^2)$ is seminormal and weakly normal (even though the residue field changed).

A useful way to construct examples is the following lemma.

Lemma 7.7. *Suppose that A is a ring, $I \subseteq$ is an ideal and B is another ring with a map $\phi : B \to A/I$. Then the pullback C of the diagram of rings*

$$\{A \to A/I \leftarrow B\}$$

has the following properties.

(i) $\operatorname{Spec} C$ *has a closed subscheme W that maps isomorphically to* $\operatorname{Spec} B$ *via the induced map (from the pullback diagram).*

(ii) *The induced map $C \to A$ gives an isomorphism between* $(\operatorname{Spec} C) \setminus W$ *and* $(\operatorname{Spec} A) \setminus (\operatorname{Spec} A/I)$*.*

(iii) *As topological spaces,* $\operatorname{Spec} C$ *is the pushout of the (dual) diagram of* Spec*'s.*

All of the examples from the previous remark can be constructed as pullbacks in this way.

There are other characterizations of weakly normal and seminormal which are of a more algebraic nature, and are often very useful. We'll only prove the second one.

Proposition 7.8. [LV81, 1.4] *Let $A \subset B$ be a finite integral extension of reduced rings; the following are then equivalent:*

(i) *A is seminormal in B*

(ii) *For a fixed pair of relatively prime integers $e > f > 1$, A contains each element $b \in B$ such that $b^e, b^f \in A$. (also see* [Ham75] *and* [Swa80] *for the case where $e = 2$, $f = 3$).*

Proposition 7.9. [RRS96, 4.3, 6.8] *Let $A \subset B$ be a finite integral extension of reduced rings where A contains $\mathbb{F}_p$ for some prime p; the following are then equivalent:*

(i) *A is weakly normal in B.*

(ii) *If* $b \in B$ *and* $b^p \in A$ *then* $b \in A$.

Proof. First we show that (i) implies (ii). Suppose, for a contradiction, that there is a $b \in B$ such that $b^p \in A$ but $b \notin A$. We will show that $A[b]$ is subintegral over A. Observe that for any element $f \in A[b]$, we know that $f^p \in A$.

First suppose that $P \in \operatorname{Spec} A$, we will show that there is exactly one prime $Q \in \operatorname{Spec} A[b]$ lying over P (obviously there is at least one and at most finitely many). Suppose that $cd \in \sqrt{P \cdot \operatorname{Spec} A[b]}$, then $(cd)^n \in P \cdot \operatorname{Spec} A[b]$ and so even better, $(cd)^{pn} \in (P \cdot \operatorname{Spec} A[b]) \cap A = P$, thus $c^p \in P$ or $d^p \in P$ by the primality of P. But if $c^p \in P$ then $c \in \sqrt{P \cdot \operatorname{Spec} A[b]}$ and likewise with d. This proves that at least the spec's line up. The residue field extensions are even easier since $A_P/P \subseteq (A[b]_{\sqrt{P \cdot A[b]_P}} / \sqrt{P \cdot A[b]_P})$ is obviously a field extension generated by a purely inseparable element (if the extension is non-trivial).

Conversely, suppose that $A \subseteq B$ is not a weakly normal extension. Thus we may assume that it is a weakly subintegral extension. Choose $b \in B$ such that $b \notin A$. It is sufficient to show that $b^{p^e} \in A$ for some $e > 0$. But first we make several reductions. Note that if condition (ii) holds on $A \subseteq B$, then it also holds after localizing at a multiplicative subset. To see this, note that if $b \in B$, $(b/s) \in S^{-1}B$ and $(b/s)^p \in S^{-1}A$, then by assumption $s^n(b/s)^p \in A$ for some n (we may assume $n = pm$). Thus $(s^{m-1}b)^p \in A$ and $s^{m-1}b \in B$ so that $s^{m-1}b \in A$ by assumption. Thus $b/s \in A$ also. Consider the locus of $\operatorname{Spec} A$ over which A is not weakly normal in B (this locus is closed – it's just the conductor of $A \subseteq B$), by localizing, we may assume that this is the maximal ideal of the local ring A. Thus $A \subseteq B$ induces a bijection on points of Spec and, all residue field extensions are trivial or purely inseparable.

Furthermore, since the extension is already both weakly subintegral and also weakly normal except at the maximal ideal, it is an isomorphism except at the maximal ideal. It follows that the residue field extension at the maximal ideal is purely inseparable. Now, consider the pull-back C of the following diagram.

$$\{B \to B/\mathfrak{m}_B \leftarrow A/\mathfrak{m}_A\}$$

This pullback C agrees with A except at the origin possibly (and by the universal property of pull-backs, we have $A \subseteq C$). However, by (ii), the extension is seminormal and since $A \subseteq C$ is clearly subintegral, must be an isomorphism. Choose $b \in B$, then $\bar{b}^{p^e} \in A/\mathfrak{m}_A$ for some $e > 0$, thus $b^{p^e} \in C$ for that same $e > 0$. $\square$

Theorem 7.10. [HR76] *If R is F-split, then it is weakly normal.*

Proof. Suppose that $r \in R^N$ and $r^p \in R$. We have the splitting $\phi : F^e_* R \to R$ which sends 1 to 1. Thus $\phi(r^p) = r\phi(1) = r$ so that $r \in R$ as well. $\square$

We now prove a partial converse in the one-dimensional case. A special case of this can be found in [GW77]. First we need a lemma.

Lemma 7.11. *If $K \subseteq L$ is a finite separable extension of fields, then any map $\phi : F^e_* K \to K$ uniquely extends to a map $\bar{\phi} : F^e_* L \to L$.*

Proof. Left to the exercises. □

Remark 7.12. In fact if $K \subseteq L$ is not separable, then the only map $F^e_* K \to K$ which extends to a map $F^e_* L \to L$ is the zero map.

Theorem 7.13. *If R one dimensional, F-finite and weakly normal with a perfect residue field, then it is F-split.*

Proof. In this proof, we will effectively classify one dimensional F-split varieties with perfect residue fields. It is harmless to assume that R is local with maximal ideal $\mathfrak{m}$ and residue field k. Let R^N denote the normalization of R. We may write $R^N = R_1 \oplus \cdots \oplus R_m$ where each R_i is a semi-local ring with maximal ideals $\mathfrak{m}_{i,1}, \ldots, \mathfrak{m}_{i,n_i}$ and residue fields $k_{i,1}, \ldots, k_{i,n_i}$ (each of which is a finite, and thus separable, extension of k).

We also have the pullback diagram

$$\{R^N = R_1 \oplus \cdots \oplus R_m \to (R_1/\mathfrak{m}_1) \oplus \cdots \oplus (R_m/\mathfrak{m}_m) = k_{1,1} \oplus \cdots \oplus k_{m,n_m} \leftarrow k\}$$

The pullback C of this diagram is an extension ring of R. It is also clearly a subintegral extension of R so $R = C$. Thus we will show that C is F-split. Choose a map $\phi : F^e_* k \to k$ that is non-zero. On each k_{i,n_i}, this map extends to a map $\phi_{i,n_i} : F^e_* k_{i,n_i} \to k_{i,n_i}$. Because each R_i is a semi-local regular ring, by Fedder's Lemma, each $\phi_{i,n_i} : F^e_* R_i/(\cap_t \mathfrak{m}_{i,t}) \to R_i/(\cap_t \mathfrak{m}_{i,t})$ extends to a map $\psi_{i,n_i} : F^e_* R_i \to R_i$. These maps then "glue" to a map on C. □

Based on the previous result, it is natural to ask whether every $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ extends to a map on the normalization? We will show that the answer is yes, but first we show a result about the conductor.

Proposition 7.14. [BK05, Exercise 1.2.E] *Given a reduced F-finite ring R with normalization R^N, the conductor ideal of R in R^N is ϕ-compatible for every $\phi \in \operatorname{Hom}_R(F^e_* R, R)$.*

Proof. The conductor ideal I can be defined as "the largest ideal $I \subseteq R$ that is simultaneously an ideal of R^N". It can also be described as

$$I := \operatorname{Ann}_R R^N/R = \{x \in R | x R^N \subseteq R\}.$$

Following the proof of [BK05, Proposition 1.2.5], consider $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. Notice, that by localization, ϕ extends to a map on the total field of fractions (which contains R^N). We will abuse notation and also call this map ϕ (since it restricts to $\phi : F^e_* R \to R$). Now choose $x \in F^e_* I$ and $r \in R^N$. Then $\phi(x) r = \phi(x r^{p^e}) \in \phi(F^e_* R) \subseteq R$. Thus $\phi(x) \in I$ as desired. □

Proposition 7.15. [BK05, Exercise 1.2.E(4)] *For a reduced F-finite ring R, every map $\phi : F^e_* R \to R$, when viewed as a map on total field of fractions, restricts to a map $\phi' : F^e_* R^N \to R^N$ on the normalization.*

Proof. We follow the hint for [BK05, Exercise 1.2.E(4)]. For any $x \in R^N \in K(R)$, we wish to show that $\phi(x) \in R^N$. First we show that we can reduce to the case of a domain. We write $R \subseteq K(R) = K_1 \oplus \cdots \oplus K_t$ as a subring of its total field of fractions. Since each minimal prime Q_i of R is ϕ-compatible, it follows that ϕ induces a map $\phi_i : F^e_* R/Q_i \to R/Q_i$ for each i. Notice that the normalization of $\operatorname{Spec} R$ is a disjoint union of components (each one corresponding to a minimal prime of R), and the ith component is equal to $\operatorname{Spec}(R/Q_i)^N$. Thus, since we are ultimately interested in ϕ' restricted to each $(R/Q_i)^N$, it is harmless to assume that R is a domain.

Suppose that I is the conductor and consider $I\phi(x)$. For any $z \in I$, $z\phi(x) = \phi(z^{p^e}x) \in \phi(F^e_* I) \subseteq I$ (notice that $z^{p^e}x \in I$ since I is an ideal of R^N). More generally, $z\phi(x)^m = z\phi(x)(\phi(x))^{m-1} \subseteq I(\phi(x))^{m-1}$ which implies that $I\phi(x)^m \subseteq I\phi(x)^{m-1}$, and so by induction $I\phi(x)^m \subseteq I \subset R$ for all $m > 0$. This implies that for every $c \in I \subseteq R$ we have that $c\phi(x)^m \in I \subseteq R$. Therefore $\phi(x)$ is integral over R by [HS06, Exercise 2.26(iv)]. □

In particular, if R is F-split, then its normalization is also F-split. This proof implies the following more general result.

Theorem 7.16. *An F-finite weakly normal one-dimensional local ring is F-split if and only if every residue field extension of $R \subseteq R^N$ is separable.*

Proof. Suppose that $(R, \mathfrak{m})$ is the local ring in question. If every residue field extension of R in R^N is separable, then the proof of Theorem 7.13 implies that R is F-split.

Conversely, if R is F-split, then there exists a surjective map $\phi : F_* R \to R$ which extends to a map $\bar{\phi} : F_* R^N \to R^N$ and which is compatible with I, the conductor ideal of $R \subseteq R^N$ (note that the induced map on R/I is non-zero). Since R is local and weakly normal, I is a radical ideal and thus $I = \mathfrak{m}$. Furthermore, I is a radical ideal on R^N and so it is a finite intersection of maximal ideals. In particular, the map ϕ restricted to $R/I = R/\mathfrak{m} = k$ extends to a map on the direct sum of its residue field extensions $R^N/I = k_1 \oplus \cdots \oplus k_n$. In particular, it extends to each k_i. But we know the map $\phi/\mathfrak{m} : F_* k \to k$ is non-zero, and since it extends to a map $F_* k_i \to k_i$ it follows that each k_i is a separable extension of k. □

8. Frobenius splittings of projective varieties and graded rings

Given a projective variety X with a ((very) ample) Cartier divisor A, we can construct the section ring

$$S := \oplus_{n \geq 0} \mathcal{O}_X(nA).$$

Likewise, given an $\mathcal{O}_X$-module $\mathscr{M}$, we can construct $M := \oplus_{n \geq 0} \mathscr{M}(n)$ where $\widetilde{M} = \mathscr{M}$ (see for example [Har77]).

If $\mathscr{L}$ is a very ample divisor corresponding to an embedding into $\mathbb{P}^n$ with associated section ring S, then S may or may not agree with the affine cone of

X (in $\mathbb{P}^n$). If X is normal, then the affine cone and $\operatorname{Spec} S$ agree if and only if the embedding is projectively normal. However, if $\mathscr{L}$ is sufficiently ample, then the two rings agree.

If X is an F-finite scheme, we can consider $F^e_* \mathcal{O}_X$ and the associated module $M := \oplus_{n \geq 0} (F^e_* \mathcal{O}_X)(n)$ and compare it with $F^e_* S$.

Question 8.1. Is M isomorphic to $F^e_* S$ as a graded S-module?

We'll answer this question with an example.

Example 8.2. Consider $X = \mathbb{P}^1_k$, $k = \bar{k}$ with the usual ample divisor $\mathcal{O}_X(1)$. In this case, $M = \oplus_{n \geq 0} F^e_* \mathcal{O}_X(np^e)$ which is quite different from $F^e_* S = \oplus_{n \geq 0} F^e_* \mathcal{O}_X(n)$ (in $F^e_* S$, some graded pieces are k-vector spaces of dimension p).

One should note that $F^e_* M$ is not a $\mathbb{Z}$-graded S-module. It is instead a $\mathbb{Z}[1/p^e]$-graded S-module. By $[F^e_* M]_{n=0 \mod \mathbb{Z}}$ we mean the direct summand of $F^e_* M$ with integral coefficients.

With this in mind.

Lemma 8.3. *Given a saturated S-module M corresponding to a coherent sheaf $\mathscr{M}$, we have an isomorphism of S-modules $[F^e_* M]_{n=0 \mod \mathbb{Z}} \cong \oplus_{n \geq 0} (F^e_* \mathscr{M})(n)$.*

This yields the following interesting result.

Proposition 8.4. *Suppose that X is an F-finite F-split scheme, and $\mathscr{L}$ is any line bundle. Then the section ring*

$$S := \oplus_{i \geq 0} H^0(X, \mathscr{L}^i)$$

is also Frobenius split.

Proof. We have the following splittings for all $i \geq 0$

$$\mathscr{L}^i \to F_* \mathscr{L}^{ip^e} \to \mathscr{L}^i$$

where the composition is an isomorphism and the first map is Frobenius. This implies that $S \to [F_* S]_{n=0 \mod \mathbb{Z}}$ splits. But $[F_* S]_{n=0 \mod \mathbb{Z}} \to F_* S$ also clearly splits. Composing these splittings gives the desired result. $\square$

The converse to the previous proposition also holds if $\mathscr{L}$ is ample.

Theorem 8.5. *Suppose that X is an F-finite F-split scheme, $\mathscr{L}$ is an ample line bundle, and S is the section ring of X with respect to $\mathscr{L}$. If S is Frobenius split, then so is X.*

We will prove this in stages. The first stage allows us to assume that $\mathscr{L}$ is (very very) ample (which isn't strictly necessary, but it is harmless and easy regardless).

Lemma 8.6. *If S is a Frobenius split graded ring, then any veronese subring is also Frobenius split.*

Proof. Suppose that $S_{(n)}$ is the nth veronese subring of S. The map $S_{(n)} \subseteq S$ is clearly split, thus $S_{(n)}$ is Frobenius split as well. $\square$

Remark 8.7. If $S_{(n)}$ is Frobenius split and p does not divide n, then S is also split as we will see later (the Veronese map is étale in codimension 1 in this case).

Lemma 8.8. *If S is a Frobenius split graded ring, then S has a "graded" Frobenius splitting.*

Proof. To define a graded Frobenius splitting, we first have to remind ourselves what the grading on F_*S is. Remember, F_*S is $\mathbb{Z}[1/p]$-graded, which makes the Frobenius map $S \to F_*S$ a degree preserving graded map. A graded splitting is thus going to be a graded (degree preserving) map $F_*S \to S$ that sends 1 to 1. Since S is split, there are obviously plenty of (possibly non-graded) maps $\phi : F_*S \to S$ which sends 1 to 1. We simply have to find a graded such map.

On the other hand, we have the *evaluation-at-1* map $\mathrm{Hom}_S(F_*S, S) \to S$. Because S is F-finite, the module $\mathbb{Z}[1/p]$-graded $\mathrm{Hom}_S(F_*S, S)$ is generated over S_0 by graded but degree shifting maps $F_*S \to S$. So suppose ϕ is an arbitrary splitting. We can write $\phi = \phi_0 + \dots + \phi_n$ where ϕ_n are degree shifting maps and ϕ_0 is degree preserving (this is a basic commutative algebra fact, a proof can be found in [BH93, Section 1.5]). It is clear that $\phi_0(1) = 1$ because $\phi(1)$ equals 1 and none of the other $\phi_i(1)$ can possibly land in the correct degree. Thus ϕ_0 is our desired degree preserving splitting. $\square$

Proof of Theorem 8.5. We may assume that $\mathscr{L}$ is very (very) ample and so our ring standard graded (generated in degree 1). We have the following composition

$$S \to [F_*S]_{n=0 \mod \mathbb{Z}} \to F_*S$$

By the previous lemma, this composition has a degree preserving graded splitting. Thus $S \to [F_*S]_{n=0 \mod \mathbb{Z}}$ also has a degree preserving graded splitting. Thus $\mathcal{O}_X = \widetilde{S} \to \widetilde{F_*S} = F_*\mathcal{O}_X$ also splits (as desired). $\square$

Finally, let us give an example to elliptic curves. We have already seen that a supersingular elliptic curve cannot be F-split (ie, an F-split elliptic curve must be ordinary), we will now prove the converse.

First we recall that $\mathscr{H}\mathrm{om}_X(F^e_*\mathcal{O}_X, \omega_X) \cong \omega_X$ (as we did this before, this was non-canonical) for X a variety over an algebraically closed (or even F-finite) field. Thus, applying the functor $\mathscr{H}\mathrm{om}_X(\_\_, \mathcal{O}_X)$ to $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ gives us a map $F^e_*\omega_X \to \omega_X$ (sometimes called the trace map).

Proposition 8.9. *Suppose that X is an ordinary [1]elliptic curve, then X is F-split.*

[1]Ordinary means that $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is injective.

Proof. We know that $H^1(X, \mathcal{O}_X) \to H^1(X, F_* \mathcal{O}_X)$ is injective. Serre-duality tells us that $H^0(X, F_* \omega_X) \to H^0(X, \omega_X)$ is surjective (where this map is induced by what we called the trace map above, one can see this via Grothendieck duality or by a degenerating spectral sequence argument). But on an elliptic curve, $\omega_X \cong \mathcal{O}_X$ so that we have a map $\phi : F_* \mathcal{O}_X \to \mathcal{O}_X$ that is surjective on global sections. In particular, there is a global section of $F_* \mathcal{O}_X$ which is sent to 1 by ϕ. This element is just a unit, and so by rescaling, we can assume that ϕ sends 1 to 1 and is thus a splitting. This means □

We also do an example of these ideas for $\mathbb{P}^n$.

Example 8.10. Suppose that $X = \mathbb{P}^n_k$ where $k = \bar{k}$. For $n = 1$ we already computed $F_* \mathcal{O}_X$. Let us at least show that $F^e_* \mathcal{O}_X$ is a direct sum of line bundles for $n > 1$ (this is an old result due to Hartshorne). Let S denote the section ring with respect to the usual $\mathcal{O}(1)$ (so that $S = k[x_0, \dots, x_n]$. We have the graded module $M := \oplus_{n \geq 0} (F^e_* \mathcal{O}_X)(n)$ which we know is a summand of $F^e_* \mathcal{O}_S$/ However, $F^e_* \mathcal{O}_S$ is a free S-module, which implies that M is projective and thus also a free S-module because M is graded (see for example [BH93, Proposition 1.5.15(d)]). So write $M = \oplus S(i)$ for various i. Therefore $F^e_* \mathcal{O}_X = \widetilde{M} = \oplus \widetilde{S(i)} = \oplus \mathcal{O}_X(i)$. In fact, the same result also holds for $F_* \mathscr{L}$ for any line bundle $\mathscr{L} = \mathcal{O}_X(n)$ on $X = \mathbb{P}^n$. The proof is the same.

We also give an example related to projective normality. Recall that on the first day of class we showed that if $Z \subseteq \mathbb{P}^n$ is compatibly Frobenius split in $X = \mathbb{P}^n$, then it's embedding is projectively normal (meaning in this case that $H^0(X, \mathcal{O}_X(i)) \to H^0(Z, \mathcal{O}_Z(i))$ is surjective for all i, this always happens for a good enough veronese). We will prove a partial converse to this statement.

Proposition 8.11. *Suppose that Z is a Frobenius split variety embedded (projectively normally) in $X = \mathbb{P}^n$. Then Z is compatibly Frobenius split in $X = \mathbb{P}^n$.*

Proof. In fact, we will show that any Frobenius splitting of Z extends to one on $\mathbb{P}^n$. Fix $\phi : F^e_* \mathcal{O}_Z \to \mathcal{O}_Z$ to be a map. This induces a graded degree-preserving map $\Phi : \oplus H^0(X, (F^e_* \mathcal{O}_Z)(i)) \to R$ on the section ring $R = \oplus H^0(X, \mathcal{O}_Z(i))$ as we've seen. However, because of the projective normality assumption, R is a quotient of $S = \oplus H^0(X, \mathcal{O}_X(i))$ (this means that the affine cone and the section ring coincide). But S is a polynomial ring and so a graded version of the proof of Fedder's Lemma implies that Φ extends to a map $\bar{\Phi} : \oplus H^0(X, (F^e_* \mathcal{O}_X)(i)) \to S$ (and we may assume that this map is also graded and degree preserving). Using the $\widetilde{\_}$ operation gives us our splitting on X which is compatible with the one on $\mathbb{Z}$. □

We'll later see that Frobenius splitting has some analog with regards to log Calabi-Yau varieties. Furthermore, I know of no analog of this statement in the log-Calabi-Yau context.

As promised, we will also attempt to describe $F_* \mathcal{O}_{\mathbb{P}^n}$.

Example 8.12. Suppose that $X = \mathbb{P}^n$. We first identify the possibly summands that can appear (as Christopher Hacon pointed out in class today), write $F_*\mathcal{O}_{\mathbb{P}^n} = \mathcal{O}_X \oplus \mathcal{O}_X(a_2) \oplus \cdots \oplus \mathcal{O}_X(a_{p^n})$ where the $a_i < 0$ are integers. Note $0 = H^n(X, \mathcal{O}_X) = H^n(X, F_*\mathcal{O}_X) = H^n(X, \mathcal{O}_X \oplus \mathcal{O}_X(a_2) \oplus \cdots \oplus \mathcal{O}_X(a_{p^n}))$.. By Serre duality this is the same as the vector space dual of $\oplus H^0(X, \mathcal{O}_X(-n-1) \otimes \mathcal{O}_X(-a_i))$. Since this is zero, none of the $-a_i$ can be larger than n (and so none of the a_i can be smaller than $-n$). In conclusion, the a_i (for $i > 1$) must all satisfy $0 > a_i \geq -n$.

We begin with $X = \mathbb{P}^2$. We know that $F_*\mathcal{O}_{\mathbb{P}^2} = \mathcal{O}_X \oplus \mathcal{O}_X(a_2) \oplus \cdots \oplus \mathcal{O}_X(a_{p^2})$ for various a_i (the rank can be computed on $\mathbb{A}^2$). We recall that on $\mathbb{P}^2$, $h^0(\mathcal{O}_X(n)) = \dim_k H^0(X, \mathcal{O}_X(n)) = \binom{n+2}{2}$. Thus $h^0((F_*\mathcal{O}_X)(1)) = h^0(\mathcal{O}_X(p)) = \binom{p+2}{2}$.

On the other hand $h^0\left(\mathcal{O}_X(1) \oplus \mathcal{O}_X(a_2+1) \oplus \cdots \oplus \mathcal{O}_X(a_{p^2}+1)\right) = \binom{3}{2} +$ the number of a_i equal to -1.

So, we consider

$$\binom{p+2}{2} - \binom{3}{2} = (p+2)(p+1)/2 - 3 = \frac{1}{2}p^2 + \frac{3}{2}p - 2.$$

In characteristic $p = 5$, the number of summands total is 25. We know that $a_1 = 0$, so there is 1 summand of the form $\mathcal{O}_X$. We also compute $\binom{p+2}{2} - \binom{3}{2} = 18$. This leaves us with 25-1-18 = 6 summands left, by our above work, these must all be equal to -2. We can also show it directly, which is what has to be done in higher dimensions.

Now we twist by (2). In this case, we have $h^0((F_*\mathcal{O}_X)(2)) = h^0(\mathcal{O}_X(2p)) = \binom{2p+2}{2} = (2p+2)(2p+1)/2$. On the other hand $\mathcal{O}_X(2) \oplus \mathcal{O}_X(1)^{\oplus 18}$ has a $\binom{2+2}{2} + ((p+2)(p+1)/2 - 3)\binom{3}{2}$ dimensional vector space of global sections. In characteristic $p = 5$, $h^0((F_*\mathcal{O}_X)(2)) = 66$ while $\binom{2+2}{2} + ((p+2)(p+1)/2)\binom{3}{2} = 60$. Thus there must be exactly 6 terms of the form $\mathcal{O}_X(-2)$.

Trying this same computation in characteristic 7 gives us the following.

- 1 copy of $\mathcal{O}_X$.
- 33 copies of $\mathcal{O}_X(-1)$.
- 15 copies of $\mathcal{O}_X(-2)$.

In general, there is

- 1 copy of $\mathcal{O}_X$.
- $\frac{1}{2}p^2 + \frac{3}{2}p - 2$ copies of $\mathcal{O}_X(-1)$.
- $\frac{1}{2}p^2 - \frac{3}{2}p + 1$ copies of $\mathcal{O}_X(-2)$.

One can check that these numbers add up to p^2.

For $X = \mathbb{P}^3$, we know that $h^0(\mathcal{O}_X(n)) = \binom{n+3}{3}$. Similar computations yield:

- 1 copy of $\mathcal{O}_X$
- $\frac{1}{6}p^3 + p^2 + \frac{11}{6}p - 3 = \frac{1}{6}(p+3)(p+2)(p+1) - 4$ copies of $\mathcal{O}_X(-1)$.
- $\frac{2}{3}p^3 - \frac{11}{3}p + 3 = \frac{1}{6}(2p+3)(2p+2)(2p+1) - (1)(10) - (\frac{1}{6}(p+3)(p+2)(p+1) - 4)(4)$ copies of $\mathcal{O}_X(-2)$

- $\frac{1}{6}p^3-p^2+\frac{11}{6}p-1 = \frac{1}{6}(3p+3)(3p+2)(3p+1)-(1)(20)-(\frac{1}{6}(p+3)(p+2)(p+1)-4)(10)-(\frac{1}{6}(2p+3)(2p+2)(2p+1)-(1)(10)-(\frac{1}{6}(p+3)(p+2)(p+1))(4))$ copies of $\mathcal{O}_X(-3)$

One again checks that the sum of these equals p^3.

I do not know of anything more general than this. It could easily be implemented into a computer if one wanted to do the check for any fixed p and n (possibly even for a generic p and fixed n). There also might be a better approach to this problem in the literature, but I didn't find it (except for the previously mentioned work of Thomsen).

9. Rational singularities

For about 40 years, rational singularities have been the gold standard of *nice* singularities. In particular, given any class of singular varieties, the first question people tend to ask is, "Does it have rational singularities?" We'll see today that rational singularities are certainly not so far from F-pure singularities.

Definition 9.1 (Watanabe)**.** Given a normal graded d-dimensional ring R with $R_0 = k$ and irrelevant ideal $\mathfrak{m} = R$, we define the *a-invariant* of R, as follows:

$$a(R) := \max\{n | (H^d_{\mathfrak{m}}(R))_n \neq 0\} = -\min\{n | (\omega_R)_n \neq 0\}.$$

Recall the following fact: If S is a standard $\mathbb{N}$-graded ring (again, you don't need standard) with irrelevant ideal $\mathfrak{m} = S_+$, with $Y = \operatorname{Proj} S$, then

$$(H^i_{\mathfrak{m}}(S))_n = H^{i-1}(Y, \mathcal{O}_Y(n)),$$

for $i > 1$. This fact is quite easy to check using Čech cohomology.

If R is an R-pure ring, then $H^d_{\mathfrak{m}}(R))_n = 0$ for $n > 0$. To see this, simply note that we have injective maps $F^e : (H^d_{\mathfrak{m}}(R))_n \to (H^d_{\mathfrak{m}}(R))_{p^e n}$ for all e and the right side vanishes for $e \gg 0$ (this is completely clear by what we wrote above in a standard graded ring by Serre vanishing). Therefore, if R is F-split, then $a(R) \leq 0$.

Watanabe also proved the following.

Theorem 9.2 (Watanabe)**.** *If R is a normal graded ring finitely generated over $k = R_0$, then R has rational singularities if and only if R satisfies the following two conditions:*

(i) *$U = \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ has rational singularities.*
(ii) *R is Cohen-Macaulay and $a(R) < 0$.*

Thus, it is obvious that there is a very close relationship between F-purity and rational singularities. Notice that I haven't defined Cohen-Macaulay or rational singularities.

9.1. **Cohen-Macaulay rings.** Briefly recall the following definition.

Definition 9.3. A local ring $(R, \mathfrak{m})$ of dimension d is called Cohen-Macaulay if there is a regular sequence of length d on R. In other words, if $x_1, \dots, x_n$ is a list of elements of $\mathfrak{m}$ such that x_{i+1} is a regular element (non-zero divisor) on $R/(x_1, \dots, x_i)$ for all i. A scheme is called Cohen-Macaulay if all of its stalks are Cohen-Macaulay local rings.

Compare the notion of a regular sequence with the (weaker) notion of a (full) system of parameters.

Definition 9.4. Elements $x_1, \dots, x_n \in R$ (a local ring of dimension n) form a *full system of parameters* if $\sqrt{x_1, \dots, x_n} = \mathfrak{m}$.

Remark 9.5. In fact, in a Cohen-Macaulay local ring, any system of parameters is a regular sequence (so if you find a system of parameters that is not a regular sequence, the ring is not Cohen-Macaulay). See [BH93].

Example 9.6. The following rings are Cohen-Macaulay.

- Any reduced one dimensional ring (choose any non-zero divisor).
- Any regular ring (any set of minimal generators of the maximal ideal will work).
- Any hypersurface singularity, or more generally, a complete intersection (this is a ring cut out by part of a regular sequence in a regular ring, and so in particular a Cohen-Macaulay ring, choose some additional parameters completing the sequence).

However, the following ring is not Cohen-Macaulay.

- $k[x, y, u, v]/((x, y) \cap (u, v)) = k[x, y, u, v]/((xu, xv, yu, yv))$. To see this, first notice that $x - u$ is not a zero divisor (it doesn't vanish on either component). Modding out by $x - u$ gives us the following ring $T := k[x, y, v]/(x^2, xv, xy, yv)$. We simply have to convince ourselves that every element of the maximal ideal of this ring is a zero divisor but this is easy since x kills every element of the maximal ideal of T.

9.2. **The Homological viewpoint on Cohen-Macaulay, Gorenstein and $\mathbb{Q}$-Gorenstein conditions.** First we remind ourselves what the derived category $D^b_{\mathrm{coh}}(X)$ is. The objects are complexes of $\mathcal{O}_X$-modules with coherent cohomology and only finitely many places with non-zero cohomology. For example, if $f : Y \to X$ is proper, then $Rf_* \mathcal{O}_Y$[2] is an object of $D^b_{\mathrm{coh}}(X)$. The morphisms of $D^b_{\mathrm{coh}}(X)$ are more complicated, they are equivalence classes of morphisms (up to chain homotopy equivalence) where we also invert all the $\mathcal{O}_X$-modules

[2] $Rf_* \mathcal{O}_Y$ is defined as follows. Take an injective resolution $\mathscr{I}^\bullet$ of $\mathcal{O}_Y$ and set $Rf_* \mathcal{O}_Y = f_* \mathscr{I}^\bullet$. A Čech resolution is fine too.

Definition 9.7. Given a scheme X, an object $\omega_X^\bullet \in D^b_{\mathrm{coh}}(X)$ is called a *dualizing complex* if it has finite injective dimension (in other words, it is quasi-isomorphic to a *FINITE* complex of injectives) and if $R\mathscr{H}\mathrm{om}^\bullet_{\mathcal{O}_X}(\omega_X^\bullet, \omega_X^\bullet) \cong \mathcal{O}_X$.

That fancy $R\,\mathrm{Hom}^\bullet_{\mathcal{O}_X}$ is some derived functor of Hom (ie, replace the second term by a complex of injectives, and apply the first operation term by term).

Generally speaking, if you have a short exact sequence, such as $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$, we do get something like a short exact sequence when applying a derived functor like Rf_* (where $f : Y \to X$ is a proper map of schemes). The output is called an exact triangle and is denoted by

$$Rf_*\mathscr{A} \longrightarrow Rf_*\mathscr{B} \longrightarrow Rf_*\mathscr{C} \xrightarrow{+1} .$$

Taking cohomology of each complex $Rf_*\mathscr{A}$, $Rf_*\mathscr{B}$ and $Rf_*\mathscr{C}$ yields the usual long exact sequence.

Remark 9.8. Dualizing complexes are unique up to shifting (you can shift any complex) and up to tensoring with invertible sheaves. See [Har66] for details.

Remark 9.9. Any quasi-projective scheme has a dualizing complex. Also, any F-finite affine scheme has a dualizing complex. If $X \subseteq \mathbb{P}^n_k$ is a projective variety, $\omega_X^\bullet$ can be defined to be $R\mathscr{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_X, \wedge^n\Omega^1_{X/k})$. For a quasi-projective variety, simply localize. Such dualizing complexes are nice because they are "normalized" at each maximal ideal of X (in particular the cohomology of $\omega_X^\bullet$ generically only lives in degree $-\dim X$). In this case $h^{-d}(\omega_X^\bullet)$ is called a *canonical module* for X and is denoted by ω_X. Again, if X is normal, then any divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$ is called a canonical divisor.

Definition 9.10. Suppose that R is a local ring with a dualizing complex $\omega_R^\bullet$ and a canonical module ω_R (for example, $R = S_\mathfrak{q}$ is the localization of a ring S that is normal and of finite type over a field k, the canonical module was constructed as $\omega_R := (\wedge^{\dim S}\Omega_{S/k})^{**}_\mathfrak{q}$).

- We say that R is *Cohen-Macaulay* if $\omega_X^\bullet$ is quasi-isomorphic to ω_X.
- We say that R is *quasi-Gorenstein*[3] if $\omega_R \cong R$ (in a non-local setting, this means that ω_X is locally free or equivalently, that K_X is a Cartier divisor).
- We say that R is $\mathbb{Q}$-*Gorenstein* if there exists an integer $n > 0$ such that nK_R is a Cartier divisor[4] (it is probably best to assume that R is normal, unless you are already familiar with the theory of Weil-divisors on non-normal varieties).
- We say that R is *Gorenstein* if it is Cohen-Macaulay and quasi-Gorenstein.

If R is not-necessarily local, we say that R is

[3]This is also sometimes called begin 1-*Gorenstein*

[4]You can have Weil divisors such that a power is a Cartier divisor, consider a ruling on the quadric cone $xy - z^2$.

Cohen-Macaulay/quasi-Gorenstein/$\mathbb{Q}$-Gorenstein/Gorenstein

if $R_{\mathfrak{q}}$ satisfies the same property for every $\mathfrak{q} \in \operatorname{Spec} R$.

Remark 9.11. Notice that $\mathbb{Q}$-Gorenstein rings are not necessarily Cohen-Macaulay (although some authors make different definitions).

Proposition 9.12. *Every regular ring is Gorenstein, and furthermore, every complete intersection is also Gorenstein (in particular, a hypersurface singularity is Gorenstein). Most generally, if R is Gorenstein/Cohen-Macaulay, then so is R/f for any regular element $f \in R$ (the converse holds locally on R).*

Proof. See for example [BH93]. $\square$

Example 9.13. The curve singularity $R = k[x, y, z]/(xy, xz, yz)$ is Cohen-Macaulay but not Gorenstein. To check that it is Cohen-Macaulay, simply notice that it is reduced and 1-dimensional. To see that it is not Gorenstein, we take a regular element $f = x+y-z$ and notice that $R/f = k[x, y]/(xy, x^2 + xy, xy + y^2) = k[x, y]/(x^2, xy, y^2)$. So we need merely check whether R/f is Gorenstein. By [BH93, Exercise 3.2.15], it is enough to find non-zero ideals I and J such that $I \cap J = 0$. But that is easy $I = (x), J = (y)$.

Finally, we also state Grothendieck duality.

Theorem 9.14. [Har66] *Given a map of schemes $f : Y \to X$ of finite type, there exists a functor $f^! : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$. If furthermore, f is proper then one has the following:*

- (i) $R\mathscr{H}om^{\bullet}_{\mathcal{O}_X}(Rf_*\mathscr{F}^{\bullet}, \mathscr{G}^{\bullet}) \cong Rf_*R\mathscr{H}om^{\bullet}_{\mathcal{O}_Y}(\mathscr{F}^{\bullet}, f^!\mathscr{G}^{\bullet})$ *where* $\mathscr{F}^{\bullet}, \mathscr{G}^{\bullet} \in D^b_{\mathrm{coh}}(X)$.
- (ii) $f^!\omega^{\bullet}_X$ *is a dualizing complex for Y (denoted now by $\omega^{\bullet}_Y$).*
- (iii) *If $f : Y \to X$ is a finite map (for example, a closed immersion), $f^!$ is identified with $R\mathscr{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \_\_)$ (viewed then as a module on Y).*

We will also use Kodaira vanishing and a relative version, Grauert-Riemenschneider vanishing.

Theorem 9.15 (Kodaira Vanishing)**.** *Suppose that X is a smooth variety of characteristic zero and $\mathscr{L}$ is an ample line bundle on X. Then $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for $i > 0$ or dually, $H^i(X, \mathscr{L}^{-1}) = 0$ for $i < \dim X$.*

Theorem 9.16. [GR70] *Suppose that $\pi : \widetilde{X} \to X$ is a proper map of algebraic varieties in characteristic zero with $\widetilde{X}$ smooth. Then $R^i\pi_*\omega_{\widetilde{X}} = 0$ for $i > 0$.*

Remark 9.17. Both of these theorems *FAIL* in characteristic $p > 0$.

9.3. **The Cohen-Macaulay and Gorenstein conditions for section rings.** To illustrate these previous notions, let us consider section rings of projective varieties with respect to ample divisors. Throughout this section, X will denote a smooth[5] projective variety over an algebraically closed field of characteristic 0 also with canonical divisor K_X. Let A be a (very (very)) ample divisor on X (ample is actually fine, but it is harmless to make it more ample for the purposes of the examples in this section).

Let $S = \oplus H^0(X, \mathcal{O}_X(nA))$ denote the section ring of S with respect to A and suppose that $\mathfrak{m} = S_+$ is the irrelevant ideal. If $Y = \operatorname{Spec} S$, then $U = \operatorname{Spec} S \setminus V(\mathfrak{m})$ is a k^*-bundle over $q : U \to X$ (far from the trivial bundle though). If S is generated in degree one, this is an easy exercise, for the more general case see for example [HS04]. We use $i : U \to Y$ to denote the inclusion.

Thus given any divisor D on X, $\oplus H^0(X, \mathcal{O}_X(D + nA))$ is the sheaf corresponding to a divisor on Y. In fact, it corresponds to the divisor q^*D extended in the unique way over the irrelevant point of Y (in other words, it corresponds to $i_* q^* D$). We use D_Y to denote this corresponding divisor on Y and make the easy observation that $n(D_Y) = (nD)_Y$. What's more important, is that $\oplus H^0(X, \mathcal{O}_X(K_X + nA))$ *IS* the canonical module ω_Y of Y (this basically follows from what we've described since q is just a k^*-bundle).

Let us first consider what this means for the quasi-Gorenstein and $\mathbb{Q}$-Gorenstein conditions. Since $\omega_Y = \mathcal{O}_Y(lK_Y)$ is a graded S-module, it will be free if and only if $\mathcal{O}_Y(lK_Y)$ is a locally free graded module (which means if and only if $\mathcal{O}_Y(nK_Y)$ is a line bundle). The graded line bundles on S are just S with a shift. In summary

Lemma 9.18. *S is quasi-Gorenstein if and only if $K_X \sim nA$ for some integer n (possibly equal to zero). Furthermore, S is $\mathbb{Q}$-Gorenstein if and only if $mK_X \sim nA$ for some integers n, m not both zero.*

Proof. We first prove the second statement which is slightly harder than the first statement. If S is $\mathbb{Q}$-Gorenstein, then $\oplus_k H^0(X, \mathcal{O}_X(mK_X + kA))$ is isomorphic to $S(n)$ for some integer n. But $\mathcal{O}_X(mK_X)$ is completely determined as an $\mathcal{O}_X$-module by $\oplus_k H^0(X, \mathcal{O}_X(mK_X + kA))$ and if it is isomorphic to $S(n) = \oplus_k H^0(X, \mathcal{O}_X(nA + kA))$, then $nA \sim mK_X$ as desired. The converse simply reverses this. □

Corollary 9.19. *If X is such that $K_X \sim 0$, then for any section ring S, S is quasi-Gorenstein.*

Remark 9.20. We also see that it is possible that for some A the section ring is $\mathbb{Q}$-Gorenstein, while for other A the section ring of the same variety is not $\mathbb{Q}$-Gorenstein. Furthermore, there are varieties with no section ring (with respect to an ample divisor) being $\mathbb{Q}$-Gorenstein.

Using something called local duality, the Cohen-Macaulay condition can also be translated as follows (even for non-graded local rings).

[5]Large parts of the section also work if X is normal, and all the results of the section hold if one assumes that X has rational singularities.

Lemma 9.21. *Suppose that* $(S, \mathfrak{m})$ *is a local ring. Then*

- S *is* Cohen-Macaulay *if and only if* $H^i_{\mathfrak{m}}(S) = 0$ *for* $i < \dim S$.

Remark 9.22. Literally, local duality says that the complex $R\Gamma_{\mathfrak{m}}(S)$ is dual to the complex $\omega_S^{\bullet}$.

If we are working with a normal section ring as before, then $H^0_{\mathfrak{m}}(S) = H^1_{\mathfrak{m}}(S) = 0$ (the first follows from the fact that S is reduced, the second from the fact that S is normal, see for example [Har77, Chapter III, Exercise 3.4]). Therefore, to show the Cohen-Macaulay condition, we only need to show the vanishing of the higher $H^i_{\mathfrak{m}}(S)$ for $1 < i \leq \dim S - 1 = \dim X$. As noted before, $(H^i_{\mathfrak{m}}(S))_n = H^{i-1}(X, \mathcal{O}_X(n))$ and so we have the following:

Lemma 9.23. *A section ring* S *of a projective variety* X *is Cohen-Macaulay if and only if* $H^j(X, \mathcal{O}_X(n)) = 0$ *for* $0 < j < \dim X$ *and all* $n \geq 0$.

Proof. The Cohen-Macaulay condition certainly implies the vanishing by the discussion above. Furthermore $H^j(X, \mathcal{O}_X(n)) = 0$ for $n < 0$ by Kodaira-vanishing (at least if X is smooth although Kodaira vanishing also holds for rational singularities) which proves the converse. □

One should thus note that it is possible that some section rings of a projective variety can fail to be Cohen-Macaulay, while others are Cohen-Macaulay (take a very high Veronese embedding). In particular, X has a section ring that is Cohen-Macaulay if and only if $H^j(X, \mathcal{O}_X) = 0$ for all $0 < j < \dim X$.

Watanabe's definition of rational singularities also can be restated as follows. Recall that he said that S has rational singularities if and only if S is Cohen-Macaulay and $a(S) < 0$ where $a(S) := \max\{n | (H^{\dim S}_{\mathfrak{m}}(S))_n \neq 0\}$.

Lemma 9.24. *A section ring* S *of a projective variety* X *has rational singularities if and only if* $H^j(X, \mathcal{O}_X(n)) = 0$ *for* $0 < j \leq \dim X$ *and all* $n \geq 0$.

Again, it is possible for some section rings to have rational singularities while other section rings do not have rational singularities.

We conclude with an example of a ring that is quasi-Gorenstein but not Cohen-Macaulay.

Example 9.25. Suppose that X is an Abelian surface (for example, the product of two elliptic curves). The irregularity of X is defined to be $\dim H^1(X, \mathcal{O}_X)$ and it an exercise in Hartshorne ([Har77, Chapter II, Section 8, Exercise 8.3(c)]) which shows that the irregularity is 2 (and in particular, non-zero).

9.4. **A definition of rational singularities.** Now we define rational singularities as well as resolutions of singularities.

Definition 9.26. Let X be a reduced scheme of (essentially) finite type over a field. We say that a map $\pi : \widetilde{X} \to X$ is a *resolution of singularities* if the following conditions are satisfied:

- (1) $\widetilde{X}$ is [regular / smooth], these notions agree in characteristic zero.

- (2) π is proper.
- (3) π is birational.

Remark 9.27. Resolutions of singularities exist in characteristic zero, [Hir64], [BM97], [BEV05], [Wło05], [Kol07]. Furthermore, there always exists a resolution satisfying the following properties.

- (a) π is projective, in other words, it is the blow-up of some (horrible) ideal.
- (b) π is an isomorphism on the locus where X is regular.
- (c) π is obtained by a sequence of blow-ups at smooth subvarieties (if $X \subseteq Y$ and Y is smooth, one may instead require that π is obtained by a sequence of blow-ups at smooth points of Y).
- (d) The reduced exceptional locus of π is a divisor with simple normal crossings (it looks analytically like $k[x_1, \ldots, x_n]/(\text{some product of the } x_i)$).

Now we define rational singularities.

Definition 9.28. A reduced local ring $(R, \mathfrak{m})$ of characteristic zero is said to have rational singularities if, for a given (equivalently any) resolution of singularities $\pi : \widetilde{X} \to X$, we have the following two conditions.

(i) $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ (in other words, X is normal)
(ii) $R^i \pi_* \mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$.

Proposition 9.29. *If X has rational singularities and $\pi : \widetilde{X} \to X$ is a resolution of singularities, then for any line bundle (or vector bundle) $\mathscr{L}$ on X, we have $H^i(X, \mathscr{L}) = H^i(\widetilde{X}, \pi^* \mathscr{L})$. In other words, cohomology of line bundles can be computed on a resolution.*

Proof. By the projection formula, $R^j \pi^* \mathscr{L} = 0$ for all $j > 0$. The statement then follows from the E_2 degeneration of the associated spectral sequence. ☐

10. Other characterizations of rational singularities

Reinterpreting the rational singularities condition in the derived category gives us the following.

Definition 10.1. If X is a singular variety and $\pi : \widetilde{X} \to X$ is a resolution, then X has *rational singularities* if and only if $\mathcal{O}_X \to R\pi_* \mathcal{O}_{\widetilde{X}}$ is an isomorphism.

We will now apply Grothendieck duality to this definition. Consider the map $\mathcal{O}_X to R\pi_* \mathcal{O}_{\widetilde{X}}$. Apply the duality functor $R\mathscr{H}om_{\mathcal{O}_X}(\_\_, \omega_X^{\bullet})$. This gives us a map

$$R\pi_* \omega_{\widetilde{X}}^{\bullet} \cong R\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \pi^! \omega_X^{\bullet}) \cong R\mathscr{H}om_{\mathcal{O}_X}(R\pi_* \mathcal{O}_{\widetilde{X}}, \omega_X^{\bullet}) \to R\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^{\bullet}) \cong \omega_X^{\bullet}.$$

Now, because $\widetilde{X}$ is smooth, it is Gorenstein so $\omega_{\widetilde{X}}^{\bullet} = \omega_{\widetilde{X}}[\dim X]$. Grauert-Riemenschneider vanishing tells us then that $R^i \pi_* \omega_{\widetilde{X}}^{\bullet} = R^{i+d} \pi_* \omega_{\widetilde{X}} = 0$ for $i + d \neq 0$ or equivalently for $i \neq -d$. If X has rational singularities, we immediately see that $h^i\omega_X^{\bullet} = R^i\pi_*\omega_{\widetilde{X}}^{mydot} = 0$ for $i \neq -d$. Thus X is Cohen-Macaulay. Conversely, we also obtain the following characterization of rational singularities due to Kempf.

Lemma 10.2. [KKMSD73] *With the notation as above, X has rational singularities if and only if X is Cohen-Macaulay and $\pi_*\omega_X \cong \omega_X$.*

Remark 10.3. One always has an inclusion $\pi_*\omega_{\widetilde{X}} \subseteq \omega_X$ so in general, one only needs to check the surjectivity.

It is a standard exercise to show that $\pi_*\omega_X = \omega_X$ in a regular ring (all the coefficients in the relative canonical divisor are positive). Once you have this, you see that the definition of rational singularities is independent of the resolution.

We'll do a standard example of rational (and non-rational) singularities in the graded case, then we'll explore some consequences of Kempf's criterion for rational singularities.

Example 10.4. Consider the (graded) ring $R = k[x, y, z]/(x^n + y^n + z^n)$. We'll set $Y = \operatorname{Spec} k[x, y, z]$ with closed subscheme $X = \operatorname{Spec} R$. We notice that the singularities of X can be resolved by blowing-up the cone-point of X (maximal ideal of R) which is the origin of Y, yielding $\pi : \widetilde{Y} \to Y$ (with exceptional $\mathbb{P}^2 = E$) which restricts to $\pi : \widetilde{X} \to X$ (with exceptional curve C). Because X is a hypersurface it is Cohen-Macaulay, and so we need to show that $\pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \cong \mathcal{O}_X(K_X)$. One can always assume that K_X and $K_{\widetilde{X}}$ agree where π is an isomorphism and furthermore, that $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ and $\mathcal{O}_Y(K_Y) = \mathcal{O}_Y$ since X is a hypersurface in $Y = \mathbb{A}^n$. Thus, we need to compute $K_{\widetilde{X}/X} = K_{\widetilde{X}} = K_{\pi|_{\widetilde{X}}}$ the relative canonical divisor of $\pi|_{\widetilde{X}}$. If this divisor is effective, then $\pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \mathcal{O}_X$ (what sections of $\mathcal{O}_X$ have poles along a divisor at a point). If it's not effective, then $\pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \subsetneq \mathcal{O}_X$ since now we are requiring sections to vanish to some order at the maximal ideal.

We know the relative canonical divisor of π though, it's simply $\mathcal{O}_{\widetilde{Y}}(2E)$ by [Har77, Chapter II, Exercise 8.5(b)]. By the adjunction formula, $(2E+\widetilde{X})|_{\widetilde{X}} = (K_Y+\widetilde{X})|_{\widetilde{X}} = K_{\widetilde{X}}$. On the other hand, we know that $(nE+\widetilde{X})|_{\widetilde{X}} = \pi^*X|_{\widetilde{X}} \sim 0$ on $\widetilde{X}$. Thus, $K_{\widetilde{X}} \sim (2-n)C$ since $E|_{\widetilde{X}} = C$.

As an easy consequence, we see that X has rational singularities if and only if $n = 1, 2$ and otherwise does not have rational singularities. Recall that the same singularity had F-split singularities if and only if $n = 1 \mod 3$.

Remark 10.5. You might ask where the adjunction formula comes from? If you have a hypersurface H on a Cohen-Macaulay variety X (if X is normal, the same statement holds because one can restrict to the Cohen-Macaulay locus which agrees with X outside a set of codimension 3), then we have a short exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

Applying $R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\_\_, \omega_X^{\bullet})$ gives us

$$\omega_H^{\bullet} = R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_H, \omega_X^{\bullet}) \to \omega_X^{\bullet} \to \omega_X^{\bullet}(H) \to \ldots$$

If we take cohomology, we get

$$0 \to \omega_X \to \omega_X(H) \to \omega_H \to h^{-\dim X+1}(\omega_X^{\bullet}) = 0$$

If X is also normal, this is exactly the statement $K_X|_H = K_H$.

First, we look at Boutot's theorem (remember, we already showed that a summand of an F-split ring is always F-split).

Theorem 10.6. [Bou87] *If $R \subseteq S$ is an extension of normal domains such that R is a direct sum of S, then if S has rational singularities, so does R.*

Proof. We first claim that there exists resolutions of singularities $\alpha : \widetilde{X} \to X = \operatorname{Spec} X$ and $\beta : \widetilde{Y} \to Y = \operatorname{Spec} S$ making a commutative diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\alpha"'] & \widetilde{Y} \arrow[l, "\gamma"'] \arrow[d, "\beta"] \\ X & Y \arrow[l, "\delta"] \end{tikzcd}$$

To see this, first resolve the singularities of X by a blow-up of an ideal, and then blow-up the extension of that ideal on Y (giving $Y' \to Y$, that will give you a diagram) and then further resolve the singularities of Y'. If we write down the derived category version of this diagram, we get

$$\begin{tikzcd} R\alpha_* \mathcal{O}_{\widetilde{X}} \arrow[r] & R\beta_* \mathcal{O}_{\widetilde{Y}} \\ \mathcal{O}_X \arrow[u] \arrow[r] & \mathcal{O}_Y \arrow[u, "f"'] \end{tikzcd}$$

This gives us the following composition:

$$\mathcal{O}_X \to R\alpha_* \mathcal{O}_{\widetilde{X}} \to R\beta_* \mathcal{O}_{\widetilde{Y}} \cong \mathcal{O}_Y \to \mathcal{O}_X$$

which is an isomorphism. Thus $\mathcal{O}_X \to R\alpha_* \mathcal{O}_{\widetilde{X}}$ splits (has a left/right inverse) in the derived category. One should note that $R\beta_* \mathcal{O}_{\widetilde{Y}}$ (or even S) is not necessarily in $D^b_{\mathrm{coh}}(X)$, simply because the map $R \subseteq S$ may not be finite / proper. They do live in $D^b(X)$ though. However, $D^b_{\mathrm{coh}}(X)$ is a full subcategory of $D^b(X)$ (see [Har66]) so we may still assume our splitting lives in $D^b_{\mathrm{coh}}(X)$.

Therefore, our result follows once we prove the following lemma.

Lemma 10.7. [Kov00] *With notation as above, if $\mathcal{O}_X \to R\alpha_* \mathcal{O}_{\widetilde{X}}$ splits in the derived category, then X has rational singularities.*

Proof. We will use Kempf's criterion for rational singularities. By assumption, we have a composition (which is an isomorphism)

$$\mathcal{O}_X \to R\alpha_* \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X$$

Applying $R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\_\_, \omega_X^\bullet)$ we obtain the following composition (which is also an isomorphism in the derived category)

$$
\begin{tikzcd}
R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^\bullet) = \omega_X^\bullet & R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(R\alpha_* \mathcal{O}_{\widetilde{X}}, \omega_X^\bullet) \arrow[l] \arrow[d, equal] & \omega_X^\bullet \arrow[l] \\
& R\alpha_* R\mathscr{H}\mathrm{om}_{\mathcal{O}_{\widetilde{X}}}(R\alpha_* \mathcal{O}_{\widetilde{X}}, \omega_{\widetilde{X}}^\bullet) \arrow[d, equal] & \\
& R\alpha_* \omega_{\widetilde{X}}^\bullet \arrow[d, equal] & \\
& \alpha_* \omega_{\widetilde{X}}[\dim X] &
\end{tikzcd}
$$

Thus $h^{-\dim X + i}\omega_X^\bullet = 0$ for $i \neq 0$, which implies that X is Cohen-Macaulay. On the other hand, taking cohomology at the $-\dim X$ place gives us

$$\omega_X \leftarrow \alpha_* \omega_{\widetilde{X}} \leftarrow \omega_X$$

where the left-most arrow is the natural inclusion (which is always injective). The fact that the composition is an isomorphism implies that the left-most arrow is also injective, and thus an isomorphism. $\square$

$\square$

11. Deformations of F-split and rational singularities.

One very fundamental property of rational singularities is the fact that they behave well in families. In fact, one also has the (a-priori) more general statement. We will prove it because eventually we will try to mimic it in characteristic $p > 0$.

First we need a finer version of resolution of singularities.

Definition 11.1. With X a reduced scheme and $Z \subset X$ any scheme (reduced or not), we say that a resolution of singularities $\pi : \widetilde{X} \to X$ is a *log resolution of* $Z \subseteq X$ if in addition we assume.

- (i) $\mathscr{I}_Z \cdot \mathcal{O}_{\widetilde{X}}$ is a invertible sheaf. In other words, it is equal to $\mathcal{O}_{\widetilde{X}}(-G)$.
- (ii) $\operatorname{exc} \pi \cup \operatorname{Supp}(G)$ is a divisor with simple normal crossings.

Remark 11.2. Log resolutions also exist in characteristic zero. Again, they also may be taken to satisfy the following properties.

- (a) π is projective, in other words, it is the blow-up of some (horrible) ideal.
- (b) π is an isomorphism on $X \setminus (\operatorname{Sing}(X) \cup Z)$.
- (c) π is obtained by a sequence of blow-ups at smooth subvarieties (if $X \subseteq Y$ and Y is smooth, one may instead require that π is obtained by a sequence of blow-ups at smooth points of Y).

Theorem 11.3. [Elk78] *Suppose that R is a local ring and that $f \in R$ is a regular element such that R/f has rational singularities, then R also has rational singularities.*

Proof. Note that since R/f is rational, R/f and thus R is Cohen-Macaulay. Let $\pi : \widetilde{X} \to X = \operatorname{Spec} R$ be a resolution of X that is also simultaneously a resolution of $H = \operatorname{Spec} R/f$. Let $\overline{H}$ be the total transform of H (that is, $\overline{H}$ is the scheme defined by $f\mathcal{O}_{\widetilde{X}}$) and let $\widetilde{H}$ denote the strict transform of H. Note, there is a natural inclusion of schemes $\widetilde{H} \to \overline{H}$. Consider the following diagram.

$$\begin{tikzcd}
 & & & \pi_*\omega_{\widetilde{H}} \arrow[d] & \\
0 \arrow[r] & \pi_*\omega_{\widetilde{X}} \arrow[r, "\times f"] \arrow[d, hook, "\psi"] & \pi_*\omega_{\widetilde{X}} \arrow[r] \arrow[d, hook, "\psi"] & \pi_*\omega_{\overline{H}} \arrow[r] \arrow[d, two heads, "\phi"] & 0 \\
0 \arrow[r] & \omega_X \arrow[r, "\times f"] & \omega_X \arrow[r] & \omega_H \arrow[r] & 0
\end{tikzcd}$$

The bottom row is exact because H is Cohen-Macaulay. The top row is exact by Grauert-Riemenschneider vanishing, [GR70]. The map labeled ϕ is surjective since the vertical composition from $\pi_*\omega_{\widetilde{H}}$ is an isomorphism. It is then enough to show that ψ is surjective.

Let C be the cokernel of ψ. The fact that ϕ is surjective means that $C \xrightarrow{\times f} C$ is surjective by the snake lemma. But this contradicts Nakayama's lemma, completing the proof. □

One can ask the slight different (a priori) question of whether rational singularities actually deform in families. In other words, given a flat family $X \to C$ over a smooth curve C, such that one fiber has rational singularities, do the nearby fibers also have rational singularities? In order to answer this, we will first need a lemma.

Lemma 11.4. *Suppose that $X/k = \bar{k}$ has rational singularities and $H \subseteq X$ is a general member of a base-point free linear system (or is simply defined by a sufficiently general equation) on X. Then H also has rational singularities.*

Proof. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities. Let $\widetilde{H}$ denote the strict transform of H (because H is general, $\widetilde{H} = \pi^{-1}(H)$). Since the linear system on X lifts to a base-point free linear system on $\widetilde{X}$, $\widetilde{H}$ is a general member of a base-point-free linear system on $\widetilde{X}$ and thus it is smooth. We will show that $R\pi_*\mathcal{O}_{\widetilde{H}} = \mathcal{O}_H$.

We work locally and assume that $H = V(f)$ for some $f \in R$ where $X = \operatorname{Spec} R$. We first claim that $L\pi^*\mathcal{O}_H \cong \pi^*\mathcal{O}_H = \mathcal{O}_{\widetilde{H}}$, but this is easy since we have the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

Since the first two terms have trivial $L^i f^*$ for $i > 0$, so does $\mathcal{O}_H = R/f$. Thus,

$$R\pi_* \mathcal{O}_{\widetilde{H}} \cong R\pi_* L\pi^* \mathcal{O}_H \cong R\pi_*(\mathcal{O}_{\widetilde{X}} \underline{\underline{\otimes}} L\pi^* \mathcal{O}_H) \cong (R\pi_* \mathcal{O}_{\widetilde{X}}) \underline{\underline{\otimes}} \mathcal{O}_H \cong \mathcal{O}_X \underline{\underline{\otimes}} \mathcal{O}_H \cong \mathcal{O}_H,$$

as desired. $\square$

Corollary 11.5. *If $f : X \to C$ is a proper family over a curve C and a fiber $f^{-1}(c)$ has rational singularities, then so do the nearby fibers.*

Proof. By Elkik's result, X has rational singularities near $f^{-1}(c)$. Choose an open set $U \subseteq X$ containing $f^{-1}(c)$ to be such that U has rational singularities. Let $Z = X \setminus U$. Then $f(Z)$ is a set of points of C (it is closed, and doesn't contain $c \in C$). A general element of C will give a general fiber of X and that fiber will be a general element of U. Thus that fiber will have rational singularities. $\square$

In this section, we'll point out that F-split singularities need not be normal, or Cohen-Macaulay (even when they are normal). We'll also show that they don't deform. It is this failure of deformation that will lead us to the right variant of rational singularities in characteristic $p > 0$.

We've already seen that F-split singularities need not be normal (although they are pretty close to normal since they are always weakly normal). The simplest example is $k[x, y]/(xy)$ which is F-split by Fedder's criterion by not normal.

F-split singularities need not be Cohen-Macaulay either. For example, $k[x, y, u, v]/((x, y) \cap (u, v))$ is F-split (this can be verified either using Fedder's criterion or the gluing methods we used for 1-dimensional varieties). Of course, this example is not normal and so one might hope for an example of a F-split normal singularity that is not Cohen-Macaulay. We provide one here.

We'll now look at the characteristic $p > 0$ situation, but first we need to have a brief discussion about reflexive rank one sheaves.

Suppose that X is normal and integral and that D is an integral Weil divisor on X. Then $\mathcal{O}_X(D) = \{f \in K(X) | \operatorname{div}(f) + D \geq 0\}$. This sheaf is rank one (clearly) and reflexive. Reflexive in this case means one of the following equivalent definitions. A sheaf $\mathscr{M}$ on X is reflexive if:

- $\mathscr{M}^{**} := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M}, \mathcal{O}_X), \mathcal{O}_X) \cong \mathscr{M}$ via the natural map. Equivalently...
- $\mathscr{M}$ is torsion free and for any open set $i : U \subseteq X$ such that $X \setminus U$ has codimension 2 (or more), $i_* \mathscr{M}|_U = \mathscr{M}$.

Remark 11.6. The second condition allows us to treat Weil divisors like Cartier divisors by setting $U = \operatorname{reg}(X)$. Generally speaking, $\mathcal{O}_X(D) \otimes \mathcal{O}_X(F) \neq \mathcal{O}_X(D + F)$ but up to double-dual ($\_^{**}$), it is true. Furthermore, for any such U, the operation i_* induces an equivalence of categories between reflexive sheaves on U and reflexive sheaves on X.

Explicitly, a map of reflexive sheaves is an isomorphism if and only if it is an isomorphism in codimension 1.

Lemma 11.7. *If X is as above and F-finite, then a torsion-free sheaf $\mathscr{M}$ is reflexive if and only if $F^e_* \mathscr{M}$ is $\mathcal{O}_X$-reflexive.*

Proof. Choose $U \subseteq X$ such that $X \setminus U$ has codimension at least 2, U is regular, and also such that $\mathscr{M}|_U$ is locally free (X is normal, so $\mathscr{M}$ is already locally free at every codimension 1 point, whose stalks are PIDs). This also implies that $F^e_* \mathscr{M}|_U$ is also locally free as an $\mathcal{O}_U$-module since $F^e_* \mathcal{O}_U$ is a locally free $\mathcal{O}_U$-module. Now, $F^e_* \mathscr{M}$ is $\mathcal{O}_X$-reflexive if and only if $i_*(F^e_* \mathscr{M}|_U) \cong F^e_* \mathscr{M}$. But that is clearly equivalent to $i_*(\mathscr{M}|_U) \cong \mathscr{M}$ which is the same thing as saying that $\mathscr{M}$ is reflexive. $\square$

We now turn to the question of whether F-split singularities deform. We consider the following situation. Suppose that R is a local ring and $f \in R$ is a regular element. If R/f is F-split, when can we conclude that R is F-split? The easiest approach would be to show that every map $\phi : F^e_*(R/f) \to R/f$ extends to a map $\bar{\phi} : F^e_* R \to R$. So we have to ask ourselves whether this is the case. We will show it is the case when R is Gorenstein, and show it is not the case when R is not Gorenstein (even if R is Cohen-Macaulay and normal).

Lemma 11.8 is also closely related to the fact that the set of Frobenius actions on $H^d_{\mathfrak{m}}(R)$ is generated by the natural Frobenius action $F^e : H^{\dim R}_{\mathfrak{m}}(R) \to H^{\dim R}_{\mathfrak{m}}(R)$; see [LS01].

Lemma 11.8. *Suppose that R is an F-finite Gorenstein local ring. By dualizing the natural map $G : R \to F^e_* R$ (apply $\mathrm{Hom}_R(\_\_, \omega_R)$), we construct the map*

$$\Psi : F^e_* \omega_R \to \omega_R$$

By fixing any isomorphism of ω_R with R (which we can do since R is Gorenstein), we obtain a map which we also call Ψ,

$$\Psi : F^e_* R \to R.$$

This map Ψ is an $F^e_* R$-module generator of $\mathrm{Hom}_R(F^e_* R, R)$.

Proof. First note that the choices we made in the setup of the lemma are all unique up to multiplication by a unit. Therefore, these choices are irrelevant in terms of proving that Ψ is an $F^e_* R$-module generator. Suppose that ϕ is an arbitrary $F^e_* R$-module generator of $\mathrm{Hom}_R(F^e_* R, R)$, and so we can write $\Psi(\_\_) = \phi(d \cdot \_\_)$ for some $d \in F^e_* R$. Using the same isomorphisms we selected before, we can view ϕ as a map $F^e_* \omega_R \to \omega_R$. By duality for a finite morphism, we obtain $\phi^\vee : R \to F^e_* R$. Note now that $G(\_\_) = d \cdot \phi^\vee(\_\_)$. But G sends 1 to 1 which implies that d is a unit and completes the proof. $\square$

Before continuing we need the following observation.

Lemma 11.9. *Given an effective Weil divisor D in a normal affine scheme $X = \operatorname{Spec} R$, the maps $\phi : \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ which are compatible with D exactly coincide with the image of*

$$\mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e - 1)D), \mathcal{O}_X) \to \mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X).$$

Proof. Suppose we have a $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ compatible with D. In other words, $\phi(F^e_* \mathcal{O}_X(-D)) \subseteq \mathcal{O}_X(-D)$. Twisting by $\mathcal{O}_X(D)$. □

We now prove our desired extension result. A similar argument (involving local duality) was used in the characteristic $p > 0$ inversion of adjunction result of [HW02, Theorem 4.9].

Proposition 11.10. *Suppose that X is normal and $D \subseteq X$ is an effective Weil divisor which is also normal. Further suppose that D is Cartier in codimension 2 and that $(p^e - 1)(K_X + D)$ is Cartier. Then the natural map of $F^e_* \mathcal{O}_X$-modules:*

$$\Phi : \mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e - 1)D), \mathcal{O}_X) \to \mathscr{H}om_{\mathcal{O}_D}(F^e_* \mathcal{O}_D, \mathcal{O}_D).$$

induced by restriction is surjective.

Proof. The statement is local so we may assume that $X = \operatorname{Spec} R$ where R is the spectrum of a local ring. The module $\mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e - 1)D), \mathcal{O}_X) \cong F^e_* \mathcal{O}_X((1 - p^e)(K_X + D))$ which is isomorphic to $F^e_* \mathcal{O}_X = F^e_* R$ because we restricted to the local setting.

Thus the image of Φ is cyclic as an $F^e_* \mathcal{O}_D$-module which implies that the image of Φ is a reflexive $F^e_* \mathcal{O}_D$-module. Therefore, it is sufficient to prove that Φ is surjective at the codimension one points of D (which correspond to codimension two points of X). We now assume that $X = \operatorname{Spec} R$ is the spectrum of a two dimensional normal local ring and that D is a Cartier divisor defined by a local equation $(f = 0)$. Since D is normal and one dimensional, D is Gorenstein, and so X is also Gorenstein.

Consider the following diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\times f"] \arrow[d, "1 \mapsto f^{p^e-1}"'] & R \arrow[r] \arrow[d, "1 \mapsto 1"] & R/f \arrow[r] \arrow[d, "1 \mapsto 1"] & 0 \\
0 \arrow[r] & F^e_* R \arrow[r, "F^e_* \times f"] & F^e_* R \arrow[r] & F^e_*(R/f) \arrow[r] & 0.
\end{tikzcd}$$

Apply the functor $\mathrm{Hom}_R(\underline{}, \omega_R)$ and note that we obtain the following diagram of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \omega_R \arrow[r, "\times f"] & \omega_R \arrow[r] & \omega_{R/f} \cong \mathrm{Ext}^1_R(R/f, \omega_R) \arrow[r] & 0 \\
0 \arrow[r] & F^e_* \omega_R \arrow[r, "F^e_* \times f"] \arrow[u, "\alpha"] & F^e_* \omega_R \arrow[r] \arrow[u, "\beta"] & F^e_* \omega_{R/f} \cong \mathrm{Ext}^1_R(F^e_*(R/f), \omega_R) \arrow[r] \arrow[u, "\delta"] & 0
\end{tikzcd}$$

The sequences are exact on the right because R is Gorenstein and hence Cohen-Macaulay. Note that by Lemma 11.8, we see that δ and α can be viewed as $F^e_* R$-module generators of the modules $\mathrm{Hom}_{R/f}(F^e_*(R/f), R/f) \cong \mathrm{Hom}_{R/f}(F^e_* \omega_{R/f}, \omega_{R/f})$ and $\mathrm{Hom}_R(F^e_* R, R) \cong \mathrm{Hom}_R(F^e_* \omega_R, \omega_R)$ respectively. Furthermore, the map labeled β can be identified with $\alpha \circ \left(F^e_*(\times f^{p^e-1})\right)$.

But the diagram proves exactly that the map $\beta \in \mathrm{Hom}_R(F^e_* R, R)$ restricts to a generator of $\mathrm{Hom}_{R/f}(F^e_* \omega_{R/f}, \omega_{R/f})$ which is exactly what we wanted to prove. $\square$

Remark 11.11. If D is not assumed to be normal but instead assumed to be S2 and Gorenstein in codimension 1, the proof goes through without change.

Corollary 11.12. *If R is normal and $\mathbb{Q}$-Gorenstein with index not divisible by $p > 0$ and R/f is normal and F-split, then R is also F-split.*

What happens if we relax these normal and $\mathbb{Q}$-Gorenstein conditions?

Example 11.13. [Fed83] [Sin99b] Consider $R = k[u, v, y, z]/(uv, uz, z(v - y^2))$. Note that $(uv, uz, z(v-y^2)) = ((u, z) \cap (v, z)) \cap (u, v - y^2)$ and so R is not normal. We will show it is not F-pure but that there is a hypersurface through the origin that is F-pure.

First, if it was F-pure, then there would be a splitting $\phi : F^e_* R \to R$ which would induce a splitting of $k[u, v, y, z]$ (by Fedder's Lemma) and also be compatible with the minimal primes of R, (u, z), (v, z) and also $(u, v - y^2)$. All of those rings are F-split, and so that isn't a problem. However, $(v, z) + (u, v - y^2) = (u, v, y^2, z)$ isn't reduced so this is impossible.

Now, consider $R/y = k[u, v, z]/(uv, uz, zv)$ which is F-pure.

Of course, you may view this as cheating since R is not normal (although it is still Cohen-Macaulay). One can construct normal examples as well, see [Sin99b, Theorem 1.1]. We're going to abandon F-splitting for a little while now, and we'll consider the following condition.

Lemma 11.14. *If R is an F-split local ring, then the natural map $\Psi : F_* \omega_R \to \omega_R$ of Lemma 11.8 is surjective. Furthermore, if R is quasi-Gorenstein, then the converse also holds.*

Proof. If R is F-split, we have a composition which is an isomorphism $R \to F_* R \to R$. Dualizing this gives us

$$\omega_R \xleftarrow{\Psi} F_* \omega_R \longleftarrow \omega_R$$

which is also an isomorphism. Thus Ψ is surjective.

Conversely, if Ψ is surjective and R is quasi-Gorenstein, then $\omega_R \cong R$ and we have a surjective map $\Psi : F_* R \to R$. $\square$

Definition 11.15. [Fed83] A Cohen-Macaulay ring R is called F-injective if the natural map $\Psi : \omega_R \to \omega_R$ is surjective.

Remark 11.16. You might ask why he called this condition F-injective and not F-surjective? It is because Ψ is the local dual of the Frobenius map $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ on local cohomology and Ψ is surjective if and only if that map is injective. More generally, in the non-Cohen-Macaulay case, he said that R was F-injective if $h^i(F_* \omega_R^{\bullet}) \to h^i(\omega_R^{\bullet})$ is surjective for every i.

Furthermore, we have the following.

Proposition 11.17. [Fed83] *Suppose that R is Cohen-Macaulay and R/f is F-injective. Then R is F-injective.*

Proof. Consider the following diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\times f"] \arrow[d, "1 \mapsto f^{p^e-1}"'] & R \arrow[r] \arrow[d, "1 \mapsto 1"] & R/f \arrow[r] \arrow[d, "1 \mapsto 1"] & 0 \\
0 \arrow[r] & F^e_* R \arrow[r, "F^e_* \times f"] & F^e_* R \arrow[r] & F^e_*(R/f) \arrow[r] & 0.
\end{tikzcd}$$

Apply the functor $\operatorname{Hom}_R(\_\_, \omega_R)$ and note that we obtain the following diagram of short exact sequences.

$$\begin{tikzcd}
 & C \arrow[r, "\eta"] & D \arrow[r] & 0 & \\
0 \arrow[r] & \omega_R \arrow[r, "\times f"] \arrow[u] & \omega_R \arrow[r] \arrow[u] & \omega_{R/f} \cong \operatorname{Ext}^1_R(R/f, \omega_R) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F^e_* \omega_R \arrow[r, "F^e_* \times f"] \arrow[u, "\alpha"] & F^e_* \omega_R \arrow[r] \arrow[u, "\beta"] & F^e_* \omega_{R/f} \cong \operatorname{Ext}^1_R(F^e_*(R/f), \omega_R) \arrow[r] \arrow[u, "\delta"] & 0
\end{tikzcd}$$

where C and D are the cokernels of α and β respectively. Thus, $C = \omega_R/\Psi_R(F^e_* \omega_R)$ and $D = \omega_R/\Psi_R(F^e_* f^{p^e-1} \omega_R))$. We have a natural surjective map

$$\mu : D = \omega_R/\Psi(F^e_* f^{p^e-1} \omega_R)) \to \omega_R/\Psi(F^e_* \omega_R)) = C$$

and we see that $\mu \circ \eta : C \to C$ is simply multiplication by f. But η surjects and thus so does $\mu \circ \eta$. But this contradicts Nakayama's lemma. $\square$

12. F-rationality

First we do an example we didn't finish last time.

Example 12.1. Let E be an ordinary elliptic curve (we know this it is F-split) and suppose that $X = E \times_k \mathbb{P}^1$ is the trivial ruled surface over E. Let S be a section ring of X with respect to a (very) very ample divisor. We will show that S is F-split (equivalently, that X is F-split) but that S is not Cohen-Macaulay. First we show that S is not Cohen-Macaulay. It is enough to show that $H^2_{S_+}(S) \neq 0$. But, $(H^2_{S_+}(S))_0 = H^1(X, \mathcal{O}_X)$. By [Har77, Chapter V, Lemma 2.4] (basic facts about the Cohomology of ruled surfaces) imply that this is $H^1(E, \mathcal{O}_E) \neq 0$ because E is an elliptic curve. Now we need to show that X is F-split. This follows from the following easy lemma:

Lemma 12.2. *Suppose that X and Y are Frobenius split schemes of finite type over k. Then $X \times_k Y$ is also Frobenius split.*

Proof. Choose $\phi : F_* \mathcal{O}_X \to \mathcal{O}_X$ and $\psi : F_* \mathcal{O}_Y \to \mathcal{O}_Y$ both splittings (in other words, sends 1 to 1). We will construct a splitting on $X \times_k Y$. We will do it locally (but canonically) so that the splitting clearly glues. Thus assume that $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. We need to construct a splitting of the Frobenius map $F_{R \otimes_k S} : R \otimes_k S \to F_* R \otimes_k S$. Given $r \otimes s \in R \otimes_k S$, we define $\alpha(r \otimes s) = \phi(r) \otimes \psi(s)$. This map is obviously $R \otimes S$-linear, and it sends 1 to 1, it also clearly glues. $\square$

Because of this, Fedder suggested that normal, Cohen-Macaulay and F-injective might be a closer match to rational singularities than F-purity. There was some evidence for this. In particular, Fedder showed that certain classes of hypersurfaces (defined over $\mathbb{Z}$) had rational singularities over $\mathbb{C}$ if and only if for all sufficiently large $p > 0$, the singularity viewed modulo p had F-pure (equivalently, F-injective) singularities. Notice that this doesn't allow $x^3 + y^3 + z^3$ because that does not have F-pure singularities for $p = 2 \bmod 3$. Elkies has since shown that for cones over planar elliptic curves (none of which have rational singularities), they are supersingular (and thus ordinary) for infinitely many p. If you are considering cones over Calabi-Yau varieties (for simplicity, we also assume that these cones are also Cohen-Macaulay, for example a K3-surface), then the condition that $\phi : F^e_* \omega_S \to \omega_S$ is known for surfaces and open for higher dimensional varieties.

F-injective singularities still aren't quite good enough. Consider the following attempted proof at showing the Cohen-Macaulay F-injective singularities are rational (ignoring the issue of characteristic $p > 0$ reduction for now).

Not a proof. Given a resolution of singularities $\pi : \widetilde{X} \to X = \operatorname{Spec} R$, we want to show that $\pi_* \omega_{\widetilde{X}} = \omega_X$. Consider the diagram:

$$\begin{tikzcd} \pi_* F_* \omega_{\widetilde{X}} = F_* \pi_* \omega_{\widetilde{X}} \arrow[r, "\pi_* \Psi_{\widetilde{X}}"] \arrow[d, hook, "F_* \alpha"'] & \pi_* \omega_{\widetilde{X}} \arrow[d, hook, "\alpha"] \\ F_* \omega_X \arrow[r, "\Psi_X"'] & \omega_X \end{tikzcd}$$

where the horizontal maps are the natural maps dual to Frobenius. If one can show that $\pi_* \Psi_{\widetilde{X}}$ and α are surjective, then that would imply that Ψ_X is surjective. Going the other way seems hard though. The following definition was thus given which easily implies that α is surjective. $\square$

Definition 12.3. An F-finite reduced ring R is called *F-rational* if it is Cohen-Macaulay and there are no proper / non-zero submodules of ω_X stable under Ψ_X (ie, $M \subseteq \omega_X$ such that $\Psi_X(M) \subseteq M$).

Why is this definition motivated? Well, in a polynomial ring with $X = \operatorname{Spec} k[x_1, \ldots, x_n]$, Φ^e_X can be identified with the map $F^e_* k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ that sends $x_1^{p^e - 1} \ldots x_n^{p^e - 1}$ to 1 and all the other monomials to zero. Given any polynomial $f \in k[x_1, \ldots, x_n]$, we can always find a monomials m and an $e \gg 0$

such that $\Phi^e_X(mf) = 1$. Thus, there are no Φ_X-stable proper ideals in a polynomial ring.

Definition 12.4. [LT81] $X = \operatorname{Spec} R$ is said to have pseudo-rational singularities if it is Cohen-Macaulay and also for every proper birational map $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ normal, $\pi_* \omega_{\widetilde{X}} = \omega_X$.

Remark 12.5. If R does not necessarily have a dualizing complex, then another definition is used (using local cohomology modules instead of ω_X, this is tantamount with replacing R by its completion). Lipman proved that regular rings have rational singularities (and that this holds under extreme generality).

Theorem 12.6 (Smith)**.** *If R is F-rational, then R is pseudo-rational.*

Proof. This should be immediate from the diagram above. $\square$

We will show that F-rational singularities satisfy many nice properties. In particular, we will study their deformations, how they behave under summands, etc. We will also show that F-rational singularities really do coincide with rational singularities by reduction mod $p > 0$.

We have defined 3 different classes of singularities now. F-rational, F-split, and F-injective (the last one has both Cohen-Macaulay and non-Cohen-Macaulay variants). We also know that F-rational singularities are F-injective (and Cohen-Macaulay) and that F-pure singularities are F-injective (meaning $h^i(F_* \omega_R^\bullet) \to h^i(\omega_R^\bullet)$ surjects for all $i > 0$, or dually $H^i_{\mathfrak{m}}(\mathcal{O}_X) \to H^i_{\mathfrak{m}}(F_* \mathcal{O}_X)$ injects for all $i > 0$). We will now investigate the normality properties of F-injective and F-rational singularities.

Lemma 12.7. *Suppose that R is F-finite and F-rational, then R is normal.*

Proof. Without loss of generality, we may assume that R is local. Let R^N be the normalization of R. We have the following inclusion map $i : R \to R^N$. We will prove that the map is an isomorphism. R is already Cohen-Macaulay, and so it is S2, and so it by Serre's criterion for normality, we simply need to check that R is regular in codimension 1. Thus by localizing we can assume that R is a 1-dimensional ring (and thus so is R^N, which is now regular). We have the following diagram of rings.

$$\begin{array}{ccc} R^N & \xrightarrow{F_{R^N}} & F_* R^N \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle F_* i} \\ R & \xrightarrow[F_R]{} & F_* R \end{array}$$

Apply $R\operatorname{Hom}_R(\_\_, \omega_R^{\bullet})$, and then Grothendieck duality for a finite map i gives us the following dual diagram.

$$\begin{tikzcd} \omega_{R^N}^{\bullet} \arrow[d, "i^{\vee}"'] & F_* \omega_{R^N}^{\bullet} \arrow[l] \arrow[d, "F_* i^{\vee}"] \\ \omega_R^{\bullet} & F_* \omega_R^{\bullet} \arrow[l] \end{tikzcd}$$

All the rings in question are Cohen-Macaulay, so we can remove all the dots and merely work with sheaves. We simply need to show that $i^{\vee}$ is injective because an isomorphism of the induced map of dualizing complexes, will imply that the original map was an isomorphism. Now, if W is a the multiplicative system of elements not contained in any minimal prime of R, we also have the diagram

$$\begin{tikzcd} \omega_{R^N} \arrow[r, "\gamma"] \arrow[d, "i^{\vee}"'] & W^{-1}(\omega_{R^N}) \cong K(R) \arrow[d, equal] \\ \omega_R \arrow[r] & W^{-1}(\omega_R) \cong K(R) \end{tikzcd}$$

where $K(R)$ is the total field of fractions of R. We notice that ω_{R^N} is torsion-free on each irreducible component thus the map γ is injective which implies that $i^{\vee}$ is also injective. $\square$

Now we turn to F-injectivity, we do not assume that R is Cohen-Macaulay but rather that $H^i_{\mathfrak{m}}(\mathcal{O}_X) \to H^i_{\mathfrak{m}}(F_*\mathcal{O}_X)$ injects for every maximal ideal $\mathfrak{m} \in R$. Note that this condition localizes, in particular $h^i(F_*\omega^{\bullet}_{R_{\mathfrak{q}}}) \to h^i(\omega^{\bullet}_{R_{\mathfrak{q}}})$ surjecting localizes.

Lemma 12.8. *Suppose that (R, m) is a reduced local ring of characteristic p, $X = \operatorname{Spec} R$ and that $X \setminus m$ is weakly normal. Then X is weakly normal if and only if the action of Frobenius is injective on $H^1_m(R)$.*

Proof. We assume that the dimension of R is greater than 0 since the zero-dimensional case is trivial. Embed R in its weak normalization $R \subset R^{\mathrm{WN}}$ (which is of course an isomorphism outside of m). We have the following diagram of R-modules.

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, hook] \arrow[d, hook] & \Gamma(X \setminus m, \mathcal{O}_{X-m}) \arrow[r, two heads] \arrow[d, "\cong"] & H^1_m(R) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R^{\mathrm{WN}} \arrow[r, hook] & \Gamma(X^{\mathrm{wn}} \setminus m, \mathcal{O}_{X^{\mathrm{wn}}-m}) \arrow[r, two heads] & H^1_m(R^{\mathrm{WN}}) \arrow[r] & 0 \end{tikzcd}$$

The left horizontal maps are injective because R and R^{WN} are reduced. One can check that Frobenius is compatible with all of these maps. Now, R is weakly normal if and only if R is weakly normal in R^{WN} if and only if every $r \in R^{\mathrm{WN}}$ with $r^p \in R$ also satisfies $r \in R$ by Proposition 7.9.

First assume that the action of Frobenius is injective on $H^1_m(R)$. So suppose that there is such an $r \in R^{\mathrm{WN}}$ with $r^p \in R$. Then r has an image in $\Gamma(X \setminus m, \mathcal{O}_{X-m})$ and therefore an image in $H^1_m(R)$. But r^p has a zero image in $H^1_m(R)$, which means r has zero image in $H^1_m(R)$, which guarantees that $r \in R$ as desired.

Conversely, suppose that R is weakly normal. Let $r \in \Gamma(X \setminus m, \mathcal{O}_{X-m})$ be an element such that the action of Frobenius annihilates its image $\overline{r}$ in $H^1_m(R)$. Since $r \in \Gamma(X \setminus m, \mathcal{O}_{X-m})$ we identify r with a unique element of the total field of fractions of R. On the other hand, $r^p \in R$ so $r \in R^{\mathrm{WN}} = R$. Thus we obtain that $r \in R$ and so $\overline{r}$ is zero as desired. $\square$

Theorem 12.9. *Let R be a reduced F-finite ring with a dualizing complex. If R is F-injective then R is weakly normal (and thus in particular seminormal). Furthermore, R is weakly normal if and only if $H^1_{\mathfrak{q}}(R_{\mathfrak{q}}) \to H^1_{\mathfrak{q}}(F_* R_{\mathfrak{q}})$ injects for all $\mathfrak{q} \in \operatorname{Spec} R$.*

Proof. A ring is weakly normal if and only if all its localizations at prime ideals are weakly normal [RRS96, 6.8]. If R is not weakly normal, choose a prime $P \in \operatorname{Spec} R$ of minimal height with respect to the condition that R_P is not weakly normal. Apply Lemma 12.8 to get a contradiction. $\square$

Corollary 12.10. *If R is a one dimensional F-finite reduced ring, then R is weakly normal if and only if it is F-injective. In particular, if R is local and has perfect residue field, then R is weakly normal if and only if R is F-split.*

This also gives us another example of an F-injective singularity that is not weakly normal.

Example 12.11. The curve singularity corresponding to the pushout $\{\mathbb{F}_p(t)[x] \to \mathbb{F}_p(t)[x]/(x) = \mathbb{F}_p(t) \leftarrow \mathbb{F}_p(t^p)[s]\}$ is weakly normal, but not F-split, since the residue field extension over the singular point (when mapping to the normalization) is not separable.

We now return to our study of F-rationality. In the case that R is a domain, we will also show that ω_R has a unique smallest submodule stable under Φ_X.

First we need a lemma.

Lemma 12.12. *Suppose that $R \to S$ is a finite map of rings such that $\operatorname{Hom}_R(S, R)$ is isomorphic to S as an S-module. Further suppose that M is a finite S-module.*

Then the natural map

$$\operatorname{Hom}_S(M, S) \times \operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(M, R) \tag{1}$$

induced by composition is surjective.

Proof. First, set α to be a generator (as an S-module) of $\operatorname{Hom}_R(S, R)$. Suppose we are given $f \in \operatorname{Hom}_R(M, R) \cong \operatorname{Hom}_R(M \otimes_S S, R)$. We wish to write it as a composition.

Using adjointness, this f induces an element $\Phi(f) \in \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, R))$. Just as with the usual Hom-Tensor adjointness, we define $\Phi(f)$ by the following rule:

$$(\Phi(f)(t))(s) = f(t \otimes s) = f(st) \text{ for } t \in M,\ s \in S.$$

Therefore, since $\mathrm{Hom}_R(S, R)$ is generated by α, for each f and $t \in M$ as above, we associate a unique element $a_{f,t} \in S$ with the property that $(\Phi(f)(t))(\_\_) = \alpha(a_{f,t}\_\_)$.

Thus using the isomorphism $\mathrm{Hom}_R(S, R) \cong S$, induced by sending α to 1, we obtain a map $\Psi : \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_S(M, S)$ given by $\Psi(f)(t) = a_{f,t}$.

We now consider $\alpha \circ (\Psi(f))$. However,

$$\alpha(\Psi(f)(t)) = \alpha(a_{f,t}) = (\Phi(f)(t))(1) = f(t).$$

Therefore $f = \alpha \circ (\Phi(f))$ and we see that the map (1) is surjective as desired. $\square$

In particular, this yields the following corollary.

Corollary 12.13. *If $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ generates $\mathrm{Hom}_R(F^e_* R, R)$ as an R-module, then ϕ^l generates $\mathrm{Hom}_R(F^{le}_* R, R)$ as an $F^{el}_* R$-module for all $l > 0$.*

Theorem 12.14. [HH94a], [BB09] *Suppose that R is an F-finite domain and that M is a torsion-free rank one R-module with a non-zero map $\phi : F^e_* M \to M$. Then there exists a unique smallest non-zero submodule $\tau(M, \phi) \subseteq M$ which is stable under ϕ (in other words, which satisfies $\phi(F^e_* N) \subseteq N$).*

Proof. Since ϕ is non-zero and M is rank-1, ϕ is generically surjective. Choose $c \in R$ such that

(i) $\phi_c : F^e_* M_c \to M_c$ generates $\left(\mathrm{Hom}_R(F^e_* M, M)\right)_c$ as an $F^e_* R$-module.
(ii) $cM \subseteq \phi(F^e_* M)$
(iii) $M_c \cong R_c$ and $F^e_* R_c \cong F^e_* M_c$ is a free R_c-module.

Condition (i) is possible because the map of $F^e_* R$-modules

$$\langle \phi \rangle_{F^e_* R} \to \mathrm{Hom}_R(F^e_* M, M)$$

is generically surjective (since ϕ is non-zero) because $\mathrm{Hom}_R(F^e_* M, M)$ is a rank one $F^e_* R$-module. Condition (ii) and (iii) are possible since M is rank-one.

Suppose now that $N \subseteq M$ is a ϕ-stable submodule. Our immediate goal is to show that $N_c = M_c \cong R_c$. Choose a prime $\mathfrak{q} \in \operatorname{Spec} R_c$, it is enough to show that $N_\mathfrak{q} = M_\mathfrak{q} \cong R_\mathfrak{q}$. Choose $0 \neq n \in N_\mathfrak{q}$ and choose $l \gg 0$ such that $F^{le}_* n \notin \mathfrak{q} \cdot F^{le}_* M_\mathfrak{q} = F^l_*(\mathfrak{q}^{[p^e]} R_\mathfrak{q})$. By hypothesis, $F^{le}_* M_\mathfrak{q}$ is a free $R_\mathfrak{q}$-module, so that $F^l_*(M_\mathfrak{q}/\mathfrak{q}^{[p^e]})$ is also free as an $R/\mathfrak{q}$-module of the same rank. Choose elements $a_2, \dots a_k \in M_\mathfrak{q}$ such that the images of $a_1 = n, a_2, \dots, a_k$ form a basis for $F^{le}_* M_\mathfrak{q}/\mathfrak{q}^{[p^e]}$ as an $R_\mathfrak{q}/\mathfrak{q}$-module. We have a map $\gamma : \oplus_i a_i R \to F^{\leq}_* M_\mathfrak{q}$.

By Nakayama's lemma, γ is surjective. But it is a surjective map between free modules of the same rank, so it is also injective. Therefore, $a_1, a_2, \dots, a_k$ form a basis for $F^{le}_* M_\mathfrak{q}/\mathfrak{q}^{[p^e]}$ over $M_\mathfrak{q}$. In particular, by projecting onto the first coordinate, there exists a map $\psi : F^{le}_* M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ such that $\psi(F^{le}_* nR_{\mathfrak{q}}) = M_{\mathfrak{q}}$ (notice that $F^{le}_* nR_{\mathfrak{q}}$ is not the summand generated by n, but it contains it). Thus $\psi(F^{le}_* N_{\mathfrak{q}}) = M_{\mathfrak{q}}$. However, $\psi(\_\_) = \phi^l(d \cdot \_\_)$ by (i) which implies that $M_{\mathfrak{q}} \supseteq N_{\mathfrak{q}} \supseteq \phi^l(F^{le}_* N_{\mathfrak{q}}) = M_{\mathfrak{q}}$ also.

Because $N_c = M_c$, we know that $c^n M \subseteq N$ for some $n > 0$. We will show that $n = 2$ works. Choose $l \gg 0$ such that $p^{le} \geq n + 1$. Then

$$c^2 M \subseteq c\phi^l(F^{le}_* M) = \phi^l(F^{le}_* c^{p^{le}} M) \subseteq \phi^l(F^{le}_* c^n M) \subseteq \phi^l(F^{le}_* N) \subseteq N$$

as desired. We call the element c^2 a *test element* for (M, ϕ).

Finally, we construct $\tau(M, \phi)$.

$$\tau(M, \phi) := \sum_{l \geq 0} \phi^l(F^{le}_* c^2 M)$$

It is certainly non-zero, and it is contained in any ϕ-stable N by construction. This completes the proof. □

Definition 12.15. Given (M, ϕ) as above, the module $\tau(M, \phi)$ is called the *test submodule of* (M, ϕ). With $\Psi_R : F_*\omega_R \to \omega_R$, the module $\tau(\omega_R, \Psi_R) = \tau(\omega_R)$ is called the simply the *test submodule.* An element $0 \neq d \in R$ is called a *test element for* (M, ϕ) if $dM \subseteq N$ for every nonzero submodule N of M satisfying $\phi(N) \nsubseteq N$. It follows from the above proof that $c \in R$ is such that R_c is regular and $M_c \cong R_c$, then c has some power which is a test element.

If R is a ring of characteristic $p > 0$ and $\pi : \widetilde{X} \to X = \operatorname{Spec} R$ is a resolution of singularities, then philosophically, $\tau(\omega_R)$ should be the submodule corresponding to $\pi_*\omega_{\widetilde{X}}$ (this submodule is independent of the choice of resolution as pointed out in [GR70]). In particular, the same argument we use to prove that F-rational singularities were pseudo-rational, can be used to show that there is always a containment $\tau(\omega_R) \subseteq \pi_*\omega_{\widetilde{X}}$, simply consider the diagram:

$$\begin{tikzcd}
\pi_* F_* \omega_{\widetilde{X}} = F_* \pi_* \omega_{\widetilde{X}} \arrow[r, "\pi_*\Psi_{\widetilde{X}}"] \arrow[d, hook, "F_*\alpha"'] & \pi_*\omega_{\widetilde{X}} \arrow[d, hook, "\alpha"] \\
F_*\omega_X \arrow[r, "\Psi_X"'] & \omega_X
\end{tikzcd}$$

We also have the following useful fact about $\tau(M, \phi)$.

Lemma 12.16. *With $\tau(M, \phi)$ as above, $\phi(F^e_* \tau(M, \phi)) = \tau(M, \phi)$.*

Proof. Because ϕ is not zero, $\phi(F^e_* \tau(M, \phi))$ is non-zero. On the other hand, it is clearly ϕ-stable thus $\phi(F^e_* \tau(M, \phi)) \supseteq \tau(M, \phi)$ by the universal property of $\tau(M, \phi)$. However, $\phi(F^e_* \tau(M, \phi)) \subseteq \tau(M, \phi)$ by definition. □

Corollary 12.17. [Vél95] *Suppose that M is a generically rank-1 module, $\phi : F^e_* M \to M$ is R-linear and that $\tau(M, \phi) = M$. Then for any non-zero submodule $N \subseteq M$, there exists an $n > 0$ such that*

$$\phi^n(F^{ne}_* N) = M.$$

In particular, for every non-zero $c \in R$, there exists an $n > 0$ such that $\phi^n(F^{ne}_* cN) = M$.

Proof. Choose $c \in R$ such that $cM \subseteq N$. We may thus assume that $N = cM$. We will show that $\phi^n(F^{ne}_* cM) \subseteq \phi^n(F^{ne}_* cM)$ which will complete the proof since we already know that $\sum_{n>0} \phi^n(F^{ne}_* cM) = M$. Now,

$$\begin{array}{rl} & \phi^n(F^{ne}_* cM) \\ = & \phi^n(F^{ne}_* c\phi(F^e_* M)) \\ = & \phi^n(F^{ne}_* F^e_*(c^{p^e} M)) \\ = & \phi^{n+1}(F^{(n+1)e}_* c^{p^e} M) \\ \subseteq & \phi^{n+1}(F^{(n+1)e}_* cM) \end{array}$$

as desired. □

Corollary 12.18. *In an F-finite ring, the F-rational locus is open.*

Remark 12.19. I only point this out because using the historically standard definitions, this is much less obvious.

Remark 12.20. The condition of the corollary is sometimes called *strong F-rationality.*

We now try to show that F-rational singularities deform (even though we don't expect pseudo-rational singularities to deform, a problem which I believe is open in general).

Theorem 12.21. *Suppose that R is a reduced local ring and $f \in R$ is a regular element. If R/f has F-rational singularities, then R also has F-rational singularities.*

Proof. The fact that R/f is normal and Cohen-Macaulay immediately imply that R is normal and Cohen-Macaulay. Therefore, we simply have to show that $\tau(\omega_R) = \omega_R$. Choose $c \in R$ such that c is a test element for (ω_R, Ψ_R), and also for $(\omega_{R/f}, \Psi_{R/f})$.

Consider the following diagram of short exact sequences (for every $e > 0$):

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "\times f"] \arrow[d, "1 \mapsto cf^{p^e-1}"'] & R \arrow[r] \arrow[d, "1 \mapsto c"] & R/f \arrow[r] \arrow[d, "1 \mapsto \bar{c}"] & 0 \\ 0 \arrow[r] & F^e_* R \arrow[r, "F^e_* \times f"] & F^e_* R \arrow[r] & F^e_*(R/f) \arrow[r] & 0. \end{tikzcd}$$

Apply the functor $\mathrm{Hom}_R(\_\_, \omega_R)$ and note that we obtain the following diagram of short exact sequences.

$$\begin{tikzcd}
 & C \arrow[r, "\eta"] & D \arrow[r] & 0 & \\
0 \arrow[r] & \omega_R \arrow[r, "\times f"] \arrow[u] & \omega_R \arrow[r] \arrow[u] & \omega_{R/f} \cong \mathrm{Ext}^1_R(R/f, \omega_R) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \oplus_e F^e_* \omega_R \arrow[r, "F^e_* \times f"] \arrow[u, "\alpha"] & \oplus_e F^e_* \omega_R \arrow[r] \arrow[u, "\beta"] & \oplus_e F^e_* \omega_{R/f} \cong \oplus_e \mathrm{Ext}^1_R(F^e_*(R/f), \omega_R) \arrow[r] \arrow[u, "\delta"] & 0
\end{tikzcd}$$

where α is the dual map to the map $R \to F^e_* R$ that sends 1 to c, and β is the dual map to the map which sends 1 to cf^{p^e-1}. Sticking direct sums in from of the terms in the bottom row guarantees that the image of α is $\tau(\omega_R)$ and that the image of δ is $\tau_R(\omega_{R/f}) = \omega_{R/f}$ by hypothesis. Of course, the image of β is contained in $\tau_R(\omega_R)$. Thus D has a natural surjection onto $C = \omega_R/\tau(\omega_R)$. Furthermore, the composition $C \to D \to C$ is as before, multiplication by f and Nakayama's lemma implies that C is zero again. $\square$

Finally, let's also compare some of the other basic properties of F-rational singularities with those of rational singularities. In particular, we might ask if Boutot's theorem still holds?

Theorem 12.22. *Suppose that $i : R \to S$ is a finite inclusion of normal local domains that splits. Then if S is F-rational (respectively F-injective) then R is F-rational (respectively F-injective).*

Proof. We first show that R is Cohen-Macaulay (note that in either case, S is Cohen-Macaulay by hypothesis). Set $\kappa : S \to R$ to be the splitting of i. By dualizing the composition $\kappa \circ i : R \to S \to R$, we obtain

$$\begin{tikzcd}
R\,\mathrm{Hom}_R(R, \omega^\bullet_R) \arrow[d, equal] & R\,\mathrm{Hom}_R(S, \omega^\bullet_R) \arrow[l] \arrow[d, equal] & R\,\mathrm{Hom}_R(R, \omega^\bullet_R) \arrow[l] \arrow[d, equal] \\
\omega^\bullet_R & \omega^\bullet_S \arrow[l] & \omega^\bullet_R \arrow[l]
\end{tikzcd}$$

Just as in the original Boutot's theorem, we immediately obtain that R is Cohen-Macaulay since the identity $h^{-\dim R+i}\omega^\bullet_R \to h^{-\dim R+i}\omega^\bullet_R$ factors through zero for $i > 0$.

For the F-injectivity, we have things pretty easy. We know that the natural map $\omega_S \to \omega_R$ is surjective. But we also have the diagram:

$$\begin{tikzcd}
\omega_R & \omega_S \arrow[l, two heads] \\
F^e_* \omega_R \arrow[u, "\Psi_R"] & F^e_* \omega_S \arrow[l, two heads] \arrow[u, two heads, "\Psi_S"']
\end{tikzcd}$$

Since Ψ_S is surjective, Ψ_R is also surjective which implies that R is F-injective.

For F-rationality, the argument is very very similar. We write down essentially the same diagram.

$$\begin{tikzcd} \omega_R & \omega_S \arrow[l] \\ \oplus_e F^e_* \omega_R \arrow[u, "\alpha"] & \oplus_e F^e_* \omega_S \arrow[l] \arrow[u, "\beta"'] \end{tikzcd}$$

Now however, the maps α and β are Ψ_R and Ψ_S (respectively) pre-multiplied by some element $c \in R$ that is a test element for both ω_R and ω_S. As before, β is surjective which implies that α is surjective. $\square$

Remark 12.23. Without the condition that S is a *finite* extension of R, these results are false. See [Wat97].

13. Reduction to Characteristic p

Our goal over the next couple weeks is to give a proof that F-rational singularities correspond to rational singularities via reduction mod p. This is hard. We will break this up into several steps.

- Introduce reduction to characteristic $p \gg 0$.
- Modulo a really hard technical lemma, prove the theorem.
- Prove the really hard technical lemma (we might put this off a little bit).

In this section we go over the necessary prerequisites to reduce a variety to characteristic p. A good introductory reference to this theory is [HH06, 2.1]. Our primary goal is the statements needed to work with rational singularities.

Let R be a finitely generated $\mathbb{C}$ algebra. We can write $R = \mathbb{C}[x_1, \dots, x_n]/I$ for some ideal I and let S denote $\mathbb{C}[x_1, \dots, x_n]$. Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. Let $\pi : Bl_J(Y) = \widetilde{Y} \to Y$ be a strong (projective) log resolution of X in Y with reduced exceptional divisor E mapping to X (induced by blowing up an ideal J). Note that we may also assume our schemes are projective; that is, we can embed Y as an open set in some $\mathbb{P}^n$, and thus take the projective closure $\overline{X}$ of X in $\mathbb{P}^n$. We may even extend π (our embedded resolution) to $\overline{\pi} : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$, a strong (projective) log resolution of $\overline{X}$ with reduced exceptional set $\overline{E}$.

There exists a finitely generated $\mathbb{Z}$ algebra $A \subset \mathbb{C}$ (including all the coefficients of a set of generators of I and those required by the blow-up of J), a finitely generated A algebra R_A, an ideal $J_A \subset R_A$, and schemes $\widetilde{Y}_A$, $\overline{X}_A$ and E_A of finite type over A such that $R_A \otimes_A \mathbb{C} = R$, $J_A R = J$, $\overline{X}_A \times_{\operatorname{Spec} A} \mathbb{C} = \overline{X}$, $Y_A \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{C} = Y$, $\overline{E}_A \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{C} = \overline{E}$ and $E_A \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{C} = E$ with E_A effective and supported on the blow-up of J_A. We may localize A at a single element so that Y_A is smooth over A and E_A is a simple normal crossing divisor over A if desired. By further localizing A (at a single element), we may assume any finite set of finitely generated R_A modules is A-free, see for example [Hun96, 3.4] and [HR74] and we may assume that A itself is regular. We can also take any finite collection of modules, for example $R^i f_* \mathcal{O}_X$ to this mixed characteristic setting, as well as maps between these modules.

Theorem 13.1 (Generic Freeness)**.** [HR74] *Let A be a Noetherian domain and let R be a finitely generated A-algebra. Let S be a finitely generated R-algebra and let E be a finitely generated S-module. Let M be a finitely generated R-submodule E and let N be a be a finitely generated A-submodule. Let $D = E/(M+N)$. Then there is a nonzero element $a \in A$ such that D_a is a free A_a-module.*

In our particular case, we may localize so that S_A, R_A, I_A, J_A, etc. are all locally free over A, as well as the various cokernels of maps between these modules.

We will now form a family of positive characteristic models of X by looking at all the rings $R_t = R_A \otimes_A k(t)$ where $k(t)$ is the residue field of a maximal ideal $t \in T = \operatorname{Spec} A$. Note that $k(t)$ is a finite, and thus perfect, field of characteristic p. In the case where we are reducing a particular maximal (closed) point, tensoring with $k(t)$ will either give us a unique closed point in our characteristic p model (if we started over $\mathbb{C}$ as we assumed), or a possibly finite set of closed points if we began by working over some other field of characteristic zero. If we are working with a non-closed point, we will have a finite set of points of $\operatorname{Spec} R_t$ pulling back to x_A. We may also tensor the various schemes Y_A, E_A, etc. with $k(t)$ to produce a characteristic p model of an entire situation.

Example 13.2. If we let $R = \mathbb{C}[x,y,z]/(x^2+y^2+z^2)$, then we would let $A = \mathbb{Z}$, so that $S_A = A[x,y,z]$, $R_A = S_A/(x^2+y^2+z^2)$, $X_A = \operatorname{Spec} R_A$, and $Y_A = \operatorname{Spec} S_A$. An obvious resolution is just blowing up the point (x,y,z) so that is what we do in S_A as well to get $\pi_A : (\widetilde{Y}_A = \operatorname{Proj}(S_A \oplus (x,y,z)t \oplus (x,y,z)^2t^2 \oplus \ldots)) \to Y_A$. In characteristic 2, this resolution is *not* a resolution of singularities since $X_{\mathbb{Z}/2}$ isn't even reduced! However, in all other characteristics it is.

Various properties of rings that we are interested in descend well from characteristic zero. For example, smoothness, normality, being reduced, and being Cohen-Macaulay all descend well [Hun96, Appendix 1]. Specifically, R_t has one of the above properties above for an open set of maximal ideals of A if and only if $R_{(\operatorname{Frac} A)}$ has the same property (in which case so does R). Furthermore, a ring R of finite type over a field k is Cohen-Macaulay if and only if for every field extension $k \subset K$, $R \otimes_k K$ is Cohen-Macaulay [BH93, 2.1.10]. Thus R_t is Cohen-Macaulay for an infinite set of primes if and only if R is Cohen-Macaulay. Likewise, it has already been shown that if R_t is seminormal for a Zariski dense set of primes, then R is seminormal [HR76, 5.31].

Let us show that the Cohen-Macaulay condition descends to characteristic $p > 0$.

Example 13.3. Suppose that $R = S/I$ where $S = \mathbb{C}[x_1, \dots, x_n]$. We know that $R\operatorname{Hom}_S(R/I, S) \cong \omega_R^{\bullet}[-\dim S]$. We consider $h^i(R\operatorname{Hom}_S(R, S))$ for some i. We will show that this vanishes in characteristic zero if and only if it vanishes in characteristic $p \gg 0$. Choose A to be a finitely generated $\mathbb{Z}$-algebra containing all the coefficients of a set of generators $\{f_i\}$of I. Let $S_A = A[x_1, \dots, x_n]$ and set I_A to be the ideal in S_A generated by those same $\{f_i\}$. Set $R_A = S_A/I_A$. If necessary, we replace A by a localization such that all modules in sight are A-free.

We claim first that $h^i(R\operatorname{Hom}_S(R, S)) \cong h^i(R\operatorname{Hom}_{S_A}(R_A, S_A)) \otimes_A \mathbb{C}$. But this is easy, since it is the same thing as $h^i(R\operatorname{Hom}_{S_A}(R_A, S_A)) \otimes_{S_A} S$ noticing that S is a flat S_A-algebra (see for example, [Mat89, Theorem 7.11, Exercise 7.7]). Therefore, we have that $h^i(R\operatorname{Hom}_S(R, S)) \neq 0$ if and only if $h^i(R\operatorname{Hom}_{S_A}(R_A, S_A)) \neq 0$ because the latter term is A-free.

We choose $\mathfrak{p}$ to be a maximal ideal of $\mathbb{A}$ and we want to do the same thing base-changing with $k = A/\mathfrak{p}$. In particular, we need to show that

$$h^i(R\operatorname{Hom}_{S_A}(R_A, S_A)) \otimes_A k \cong h^i(R\operatorname{Hom}_{S_k}(R_k, S_k)).$$

This is more complicated because k is not A-flat. Choose a free S_k-resolution $P_{\bullet}$ of R_A, tensoring with S_k over S_A turns it into a complex mapping to R_k. Alternately, tensoring with k over A keeps it acyclic (since it would then correspond to Tor of the A-free module R_A). Thus, it is still a free-resolution of R_k. The statement then reduces to the question of whether $\operatorname{Hom}_{S_A}(\_\_, S_A) \otimes k$ is the same as $\operatorname{Hom}_{S_A}(\_\_ \otimes_A k, S_k)$ for A-free modules in the blank. Choose M to fill in the blank, an A-free S_A-module. Choose $F \to G \to M \to 0$ to be an exact sequence with F and G chosen as S_A-free modules, by localizing further, we may assume that $H = \operatorname{Image}(F) \subseteq G$ is A-free. We have a natural map

$$\gamma(F) : \operatorname{Hom}_{S_A}(F, S_A) \otimes k \to \operatorname{Hom}_{S_A}(F \otimes_A k, S_k)$$

This can also be described as

$$\operatorname{Hom}_{S_A}(F, S_A) \otimes_{S_A} S_k \to \operatorname{Hom}_{S_A}(F \otimes_{S_A} S_k, S_k)$$

But since $= S_A^n$, this is just $S_k^n \to S_k^n$ in an obvious isomorphism. Thus $\gamma(F)$ and $\gamma(G)$ are isomorphisms. Now, consider the following diagram (where all tensor products are over A)

$$\begin{tikzcd}
\operatorname{Tor}_A^1(\operatorname{Hom}_{S_A}(H, S_A), k) = 0 \arrow[r] \arrow[d] & \operatorname{Hom} S_A(M, S_A) \otimes k \arrow[r] \arrow[d] & \operatorname{Hom}_{S_A}(G, S_A) \otimes k \arrow[r] \arrow[d, "\sim"'] & \operatorname{Hom} S_A(F, S_A) \otimes k \arrow[d, "\sim"'] \\
\operatorname{Hom}_{S_A}(\operatorname{Tor}_A^1(H, k), S_k) = 0 \arrow[r] & \operatorname{Hom}_{S_A}(M \otimes k, S_k) \arrow[r] & \operatorname{Hom}_{S_A}(G \otimes k, S_k) \arrow[r] & \operatorname{Hom}_{S_A}(F \otimes k, S_k)
\end{tikzcd}$$

Therefore, if you want to determine if a ring is Cohen-Macaulay, in some sense it is sufficient to check it in characteristic $p \gg 0$.

Note that if one also has the coordinates of a point $x \in X$ (closed or not), one can reduce that closed point to characteristic p as well. Let $x \subset R$ be a prime ideal of R and simply include coefficients for a set of generators of x into

A. This gives us an ideal $x_A \in R_A$. Note without loss of generality we may assume that $x_A = x \cap R_A$ so that x_A is prime. Furthermore, we may assume that if we tensor the short exact sequence

$$0 \to x_A \to R_A \to R_A/x_A \to 0$$

by $\otimes_A \mathbb{C}$ we simply re-obtain

$$0 \to x \to R \to R/x \to 0,$$

the original exact sequence. Note that we may certainly also assume that R_A/x_A is A-free as well. In particular, if x is maximal (closed) and if we are working over $\mathbb{C}$ or any other algebraically closed field of characteristic zero, we may assume that $R_A/x_A = A$ since $R/x \cong \mathbb{C}$. Otherwise (still in the case where x is maximal) we see that R_A/x_A is a module-finite extension of A.

The following lemma is very useful for reducing cohomology to prime characteristic, the method of proof is essentially the same as [Har77, Chapter III, Section 12] (just different modules are flat).

Lemma 13.4. [Har98, 4.1] *Let X be a noetherian separated scheme of finite type over a noetherian ring A, and let $\mathscr{F}$ be a quasi-coherent sheaf on X, flat over A. Suppose that $H^i(X, \mathscr{F})$ is a flat A-module for each $i > 0$. Then one has an isomorphism*

$$H^i(X, \mathscr{F}) \otimes_A k(t) \cong H^i(X_{k(t)}, \mathscr{F}_{k(t)})$$

for every point $t \in T = \operatorname{Spec} A$ and $i \geq 0$, where $k(t)$ is the residue field of $t \in T$, $X_{k(t)} = X \times_T \operatorname{Spec}(k(t))$, and $\mathscr{F}_{k(t)}$ is the induced sheaf on $X_{k(t)}$.

Remark 13.5. In particular, by the previous lemma and flat base change, we see that $H^i(X_{\mathbb{C}}, \mathscr{F}_{\mathbb{C}}) = 0$ if and only if $H^i(X_{k(t)}, \mathscr{F}_{k(t)}) = 0$ for an open set of $t \in \operatorname{Spec} A$.

By making various cokernels of maps free A-modules, we may also assume that maps that are surjective over $\mathbb{C}$ are still surjective over A, and thus surjective in our characteristic p model as well. The following example illustrates this.

Example 13.6. Suppose we are given a scheme X with a divisor $E \subseteq X$ all over $\mathbb{C}$. Suppose that some map

$$H^i(X, \mathcal{O}_X) \to H^i(E, \mathcal{O}_E)$$

surjects for some i. Then we consider the corresponding map

$$H^i(X_A, \mathcal{O}_{X_A}) \to H^i(E_A, \mathcal{O}_{E_A}) \to C \to 0$$

with cokernel C. We may of course localize so that C is locally free over A, in which case, since tensoring with $\mathbb{C}$ over A cannot annihilate a non-zero element, we obtain that $C = 0$. Therefore the corresponding map was surjective in the first place. Then, since tensor is right exact, we apply 13.4 and obtain that the map

$$H^i(X_t, \mathcal{O}_{X_t}) \to H^i(E_t, \mathcal{O}_{E_t})$$

surjects as well.

Definition 13.7. Given a class of singularities P defined in characteristic $p > 0$, we say that a variety X in characteristic 0 has singularities of *open P-type* if for all sufficiently large choices of A as above, and all but finitely many maximal ideal $\mathfrak{p} \in A$, $X_{\mathfrak{p}}$ has P-singularities. We say that X in characteristic zero has singularities of *dense P-type* if for all sufficiently large choices of A as above, there exists a Zariski-dense set of maximal ideals $\mathfrak{p} \in \operatorname{Spec} A$ such that $X_{\mathfrak{p}}$ has P-singularities. In this way we can define singularities of (open/dense) F-rational, F-injective and F-split/pure type.

Remark 13.8. In general, the singularities we consider are stable under base change by finite field extensions, so one only needs to check a single finitely generated $\mathbb{Z}$-algebra A.

Theorem 13.9. *Suppose that X is a variety of characteristic zero. Then if X has dense F-rational type, X has rational singularities.*

Proof. Take a resolution $\pi : \widetilde{X} t\emptyset X$. The map $\omega_{\widetilde{X}} \to \omega_X$ surjects if and only if it's reduction to characteristic $p \gg 0$ does (and we've already shown that). The Cohen-Macaulay condition was done in the example above. $\square$

Let's do another example of this sort of proof. We give another definition.

Definition 13.10. Suppose that X is a normal Cohen-Macaulay variety of characteristic zero and suppose that $\pi : \widetilde{X} \to X$ is a log resolution, fix E to be the exceptional divisor. We say that X has *Du Bois singularities* if $\pi_* \omega_{\widetilde{X}}(E) = \omega_X$.

Remark 13.11. Du Bois singularities can be defined for even reduced varieties, but the definition (and proofs) are much harder.

Theorem 13.12. *Suppose that X is normal, Cohen-Macaulay and has dense F-injective type, then X has Du Bois singularities.*

Proof. Let $\pi : \widetilde{X} \to X$ be a log resolution of X with exceptional divisor E. We reduce this entire setup to characteristic $p \gg 0$ such that the corresponding X is F-injective. Let $F^e : X \to X$ be the e-iterated Frobenius map.

We have the following commutative diagram,

$$\begin{tikzcd} F^e_* \pi_* \omega_{\widetilde{X}}(p^e E) \arrow[r] \arrow[d, "\rho"'] & \pi_* \omega_{\widetilde{X}}(E) \arrow[d, "\beta"] \\ F^e_* \omega_X \arrow[r, "\phi"] & \omega_X \end{tikzcd}$$

where the horizontal rows are induced by the dual of Frobenius, $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ and the vertical arrows are the canonical maps. By hypothesis, ϕ is surjective. On the other hand, for $e > 0$ sufficiently large, the map labeled ρ is an isomorphism. Therefore the map $\phi \circ \rho$ is surjective which implies that the map β

is also surjective. But then it must have been surjective in characteristic zero as well, and in particular, X has Du Bois singularities. $\square$

Remark 13.13. The above theorem also holds without the Cohen-Macaulay and normal hypotheses, but the proof is much more difficult.

14. Rational singularities are open F-rational type

Our main goal will to be to give a proof (modulo a hard theorem) of the following.

Theorem 14.1. [Har98], [MS97] *If X is in characteristic zero has rational singularities, then it has open F-rational type.*

To prove this, we will use the following lemma which we will black-box for today. First recall that on a normal variety X, a $\mathbb{Q}$-divisor is just an element of $\mathrm{div}(X) \otimes \mathbb{Q}$, a formal sum of prime divisors.

Lemma 14.2. [Har98] *Suppose that R_0 is a ring of characteristic zero, $\pi : X_0 \to \operatorname{Spec} R_0$ is a log resolution of singularities, D_0 is a π-ample $\mathbb{Q}$-divisor with simple normal crossings support. We reduce this setup to characteristic $p \gg 0$. Then the natural map*

$$(F^e)^\vee = \Phi_{X_p} : F^e_* \omega_{X_p}(\lceil p^e D_p \rceil) \to \omega_{X_p}(\lceil D_p \rceil)$$

surjects.

Before we use this, let us explain some points. We will assume that $\pi : X_0 \to \operatorname{Spec} R_0$ is projective, and thus the blow-up of some ideal sheaf $J \subseteq R$. It follows then that $J \cdot \mathcal{O}_{X_0} = \mathcal{O}_{X_0}(-F)$ is relatively ample (here, F is an effective divisor), in particular, the relatively effective divisors are *not* effective. Our divisor D_0 will in practice to be something close to the form $-\varepsilon F$ where ε is a small negative number (actually, we may twist by a Cartier divisor, really a test element, from $\operatorname{Spec} R_0$ as well)

Thus, in our situation $\lceil D_p \rceil$

Here is the proof idea. Choose $d^n \in R$ to be a test element for ω_{R_p} (we can essentially find one of these in characteristic zero if we are clever).

For an appropriate $D = \varepsilon(-F - \mathrm{div}(d^n))$ as above, we construct a diagram:

$$\begin{tikzcd}
\pi_* F^e_* \omega_{X_p}(\lceil p^e D_p \rceil) \arrow[r] \arrow[d] & \pi_* \omega_{X_p}(\lceil D_p \rceil) \cong \pi_* \omega_{X_p} \arrow[d] \\
F^e_* d^n \omega_{R_p} \arrow[r, "\Phi_{R_p}"] & \omega_{R_p}
\end{tikzcd}$$

We know that $\Phi_{R_p}(F^e_* d\omega_{R_p})$ is contained in $\tau(\omega_{R_p}, \Phi_{R_P})$. Thus we have

$$\omega_R = \pi_* \omega_{X_p} = \pi_* \Phi_X(F^e_* \omega_{X_p}(\lceil p^e D_p \rceil)) \subseteq \Phi_R(F^e_* d\omega_{R_p}) \subseteq \omega_{R_p}$$

which completes the proof.

Let us explain how we find our $d \in R_0$ in characteristic zero. We fix d such that $(R_0)_d$ is regular. It follows that some power of d is a test element for $(\omega_{R_p}, \Phi_{R_p})$ in any characteristic. In particular, we may then choose our resolution of singularities $\pi : X_0 \to \operatorname{Spec} R_0$ such that X_0 is a log resolution of $(X, (d)^n)$ for any integer $n > 0$. We choose $-F$ as above and set $D = \varepsilon(-F - \operatorname{div}(d))$ where $\lceil D \rceil = 0$. After reducing to characteristic $p \gg 0$, find $n > 0$ such that d^n is a test element. Fix p^e such that $\varepsilon p^e \geq n$. We then claim that we have a map $\pi_* \omega_{X_p}(\lceil p^e D_p \rceil) \subseteq d\omega_R$. It is sufficient to check this in codimension 1, and so we are simply reduced to verifying that $\lceil -\varepsilon p^e \div_R (d) \lceil \leq -n \div_R (d)$ which is obvious. The proof of the theorem then follows from the result above.

In fact, the same proof gives us the more general result.

Theorem 14.3. [Har05], [Smi00b] *With the notation as above* $(\pi_* \omega_X)_p = \tau(\omega_{R_p}, \Phi_{R_p})$.

Remark 14.4. This was not obvious when it was first proved. While the proof I gave is philosophically the same, it is substantially streamlined in comparison to Hara's original proof. In particular, we avoid several applications of local duality.

Remark 14.5. For a (quasi-)Gorenstein ring R with $R \cong \omega_R$ and $\pi : X \to \operatorname{Spec} R$ as above, $\pi_* \omega_X$ is an example of a multiplier ideal (it is independent of the resolution by [GR70]). Thus the previous result says that the multiplier ideal coincides with the test ideal $\tau(R, \Phi_R)$ for quasi-Gorenstein rings.

Question 14.6. Is it true that if X has Du Bois singularities, then X has dense F-injective type?

Note that X cannot have open F-injective type by the example $k[x, y, z]/(x^3 + y^3 + z^3)$ which is F-injective if and only if $p = 1 \mod 3$.

15. Multiplier ideals, log terminal and log canonical singularities

In the past section, we found analogs of F-injective and F-rational singularities. We want to do the same for F-split singularities.

Definition 15.1. A *pair* (X, Δ) is the combined information of a normal variety X and a (usually effective) $\mathbb{Q}$-divisor Δ. We also typically assume that (X, Δ) is *log* $\mathbb{Q}$*-Gorenstein* which means that $K_X + \Delta \sim_{\mathfrak{q}} mD$ where $m \in \mathbb{Q}$ and D is a Cartier divisor (in other words, this means that $K_X + \Delta$ is $\mathbb{Q}$-Cartier). Occasionally we will also consider *triples* $(X, \Delta, \mathfrak{a}^t)$ where $\mathfrak{a}$ is an ideal sheaf on X and $t \geq 0$ is a real number (additional generalizations are also possible where $\mathfrak{a}^t$ is replaced by a graded system of ideals, or even a formal product of such ideals). For the moment, we will assume that $\mathfrak{a}$ is a

Definition 15.2. A *log resolution* $\pi : \widetilde{X} \to X$ of a pair or triple $(X, \Delta, \mathfrak{a}^t)$ is a resolution of singularities such that $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-G)$ and also with divisorial exceptional set E such that E, G and the strict transform $\pi_*^{-1}\Delta$ of Δ are all in simple normal crossings.

In this setting, we can choose divisors $K_{\widetilde{X}}$ and K_X that agree wherever π is an isomorphism. Then we can consider:

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG = \sum a_i E_i$$

or equivalently

$$K_{\widetilde{X}} + (-\sum a_i E_i) = \pi^*(K_X + \Delta) - tG$$

where the E_i are prime divisors. Here most of the E_i are effective except for those that agree with components of Δ or divisorial components of $V(\mathfrak{a})$. We should explain the term $\pi^*(K_X + \Delta)$ and note that for the purposes of this course, we will only define this when $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Set choose $0 \neq n \in \mathbb{Z}$ such that $n(K_X + \Delta)$ is Cartier. Then

$$\pi^*(K_X + \Delta) := \frac{1}{n}\pi^*\left(n(K_X + \Delta)\right)$$

The a_i that appear in the above formula are called *discrepancies.* Numbers a_i associated to an exceptional divisor E_i are called *exceptional discrepancies.*

Why might one want to do this (work with these Δ at all)?

(a) If K_X is Cartier (or $\mathbb{Q}$-Cartier, then you can pull back K_X as described above). But if not, it's much less clear how to pull back K_X, see [DH09].
(b) As one changes from one variety to another (via restriction, finite or birational maps) one can pick up a Δ even if you didn't already start with one. For example, if $\pi_* : Y \to X = \operatorname{Spec} k[x, y, z]/(x^4 + y^4 + z^4)$ is the obvious resolution of singularities, then $\pi^* K_X = K_Y + 2E$ where E is the copy of the exceptional divisor. For some purposes, it is useful to keep this information around. In particular, the data of the pair $(Y, 2E)$ may be as good as the data of X.
(c) (This is another variant of (b)) If one is compactifying a variety X, one often compactifies with a nice divisor D such that $\overline{X} \setminus D = X$. Keeping track of this D is also useful.

One can actually define some additional classes of singularities in this setting.

Definition 15.3.
- We say a triple $(X, \Delta, \mathfrak{a}^t)$ is *log canonical* (or *lc*) if all the discrepancies a_i satisfy $a_i \geq -1$. One can check this on a single log resolution.
- We say a triple $(X, \Delta, \mathfrak{a}^t)$ is *Kawamata log terminal* (or *klt*) if all the discrepancies a_i satisfy $a_i \geq -1$. One can check this on a single log resolution.
- We say that a triple $(X, \Delta, \mathfrak{a}^t)$ is *purely log terminal* (or *plt*) if all the exceptional discrepancies a_i satisfy $a_i \geq -1$ for all log resolutions. One needs a sufficiently big log resolution in order to check this.
- One can also define *canonical* and *terminal* singularities by requiring that all exceptional discrepancies satisfy $a_i \geq 0$ and $a_i > 0$ respectively.

Remark 15.4. If X is smooth, $\Delta = 0$ and $\mathfrak{a} = \mathcal{O}_X$, then all exceptional discrepancies are positive (on all log resolutions). Thus smooth varieties have terminal singularities.

Remark 15.5. Suppose that $(X, \Delta, \mathfrak{a}^t)$ is klt / lc and $\Delta' \leq \Delta$ is such that $K_X + \Delta'$ is $\mathbb{Q}$-Cartier. Then (X, Δ') is also klt/lc.

Example 15.6. If X is smooth and Δ is a $\mathbb{Q}$-divisor with simple normal crossings support, then if all the coefficients of components of Δ are less than 1, (X, Δ) is klt. If all the coefficients are less than or equal to 1, then (X, Δ) is lc. If all the coefficients are less than or equal to 1, and none of the components with coefficient 1 intersect, then (X, Δ) is plt.

Definition 15.7. Given a $\mathbb{Q}$-divisor $D = \sum b_i D_i$ where the D_i are prime divisors, we define

$$\lceil D \rceil := \sum \lceil b_i \rceil D_i \text{ and } \lceil D \rceil := \sum \lceil b_i \rceil D_i$$

Definition 15.8. With notation as above, the *multiplier ideal* $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ is defined to be

$$\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG \rceil) = \pi_* \mathcal{O}_{\widetilde{X}}(\lceil \mathcal{O}_{\widetilde{X}}(\sum a_i E_i) \rceil)$$

This always is a subsheaf of $\mathcal{O}_X$ as long as $\Delta \geq 0$ (the point is that effective exceptional divisors can be ignored when pushing down).

Much of the multiplier ideals usefulness ties in with a relative version of Kawamata-Viehweg vanishing (a generalization of Grauert-Riemenschneider vanishing, which was a relative version of Kodaira vanishing).

Theorem 15.9. [Kaw82][Vie82] $\mathbf{R}^i \pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG \rceil) = 0$ *for* $i > 0$.

Remark 15.10. If Δ is effective, we see that $(X, \Delta, \mathfrak{a}^t)$ is klt if and only if $\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \mathcal{O}_X$. Furthermore, if $(X, \Delta, \mathfrak{a}^t)$ is log canonical, then $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ is a radical ideal. Furthermore, if $(X, \Delta, \mathfrak{a}^t)$ is klt and $\Delta \geq 0$, then $\lfloor \Delta \rfloor = 0$.

Example 15.11. Consider $X = \mathbb{A}^2$ and $\Delta = \frac{2}{3} \operatorname{div}_X(xy(x - y))$. A log resolution $\pi : \widetilde{X} \to X$ can be obtained by doing one blow-up at the origin, use E to denote the exceptional divisor. We set $K_X = 0$, then

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) = K_{\widetilde{X}} - \frac{2}{3} \operatorname{div}_{\widetilde{X}}(xy(x-y)) = E - \frac{2}{3}(3E + C_1 + C_2 + C_3) = -E - \frac{2}{3}(C_1 + C_2 + C_3)$$

where the C_i are the strict transforms of the three curves in the support of Δ. Thus (X, Δ) is log canonical, but not Kawamata/purely log terminal. Furthermore, $\mathcal{J}(X, \Delta) = (x, y) = \mathfrak{m}$.

An example of a plt pair that is not klt is $(\mathbb{A}^2, \operatorname{div}(x))$. Generally speaking the pair made up of a smooth variety and a smooth divisor is always purely log terminal, but a pair made up of a smooth variety and a simple normal crossings divisor is not plt – $(\mathbb{A}^2, \operatorname{div}(xy))$ is not purely log terminal (even though it is its own log resolution).

In general, klt singularities are rational, klt singularities are log canonical, Gorenstein rational singularities are klt. Log canonical singularities are Du Bois and Gorenstein Du Bois singularities are log canonical.

Proposition 15.12. [Elk81] *If (X, Δ) is klt and $\Delta \geq 0$, then X has rational singularities. If X is Gorenstein, then if X has rational singularities, X has canonical (and thus klt) singularities.*

Proof. Let $\pi : \widetilde{X} \to X$ be a log resolution. We have a natural inclusion $\mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta)\rceil)$. Applying $R\pi_*$ gives us the composition

$$\mathcal{O}_X \to R\pi_*\mathcal{O}_{\widetilde{X}} \to R\pi_*\mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG\rceil) \cong \mathcal{J}(X, \Delta) = \mathcal{O}_X$$

This map is clearly an isomorphism in codimension 1, and so it is an isomorphism. Thus $\mathcal{O}_X \to R\pi_*\mathcal{O}_{\widetilde{X}}$ splits, and so X has rational singularities.

In the Gorenstein case, for the converse direction, if $\omega_X \cong R\pi_*\omega_{\widetilde{X}}$, then $\mathcal{O}_X \cong R\pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} - \pi^*K_X)$. ☐

Proposition 15.13. [KK09] *If (X, Δ) is log canonical, then X has Du Bois singularities.*

Proof. We only provide a proof in the Cohen-Macaulay case (which is the only case where we defined Du Bois singularities). Set $\pi : \widetilde{X} \to X$ to be a log resolution with reduced exceptional divisor E. There exists a natural inclusion $\iota : \varrho_*\omega_{X'}(G) \subseteq \omega_X$, so the question is local. We may assume that X is affine and need only prove that every section of ω_X is already contained in $\varrho_*\omega_{X'}(G)$.

Next, choose a canonical divisor $K_{X'}$ and let $K_X = \varrho_*K_{X'}$. As $\Delta' = \varrho_*^{-1}\Delta$, it follows that the divisors $K_{X'} + \Delta'$ and $\varrho_*^{-1}(K_X + \Delta) = \Delta'$ may only differ in exceptional components. We emphasize that these are actual divisors, not just equivalence classes (and so are B and B').

Since X and X' are birationally equivalent, their function fields are isomorphic. Let us identify $K(X)$ and $K(X')$ via ρ^* and denote them by K. Further let $\mathscr{K}$ and $\mathscr{K}'$ denote the K-constant sheaves on X and X' respectively.

Now we have the following inclusions:

$$\Gamma(X, \varrho_*\omega_{X'}(E)) \subseteq \Gamma(X, \omega_X) \subseteq \Gamma(X, \mathscr{K}) = K,$$

and we need to prove that the first inclusion is actually an equality. Let $g \in \Gamma(X, \omega_X)$. So

$$0 \leq \operatorname{div}_X(g) + K_X \leq \operatorname{div}_X(g) + K_X + \Delta \tag{2}$$

As (X, Δ) is log canonical, there exists an $m \in \mathbb{N}$ such that $mK_X + m\Delta$ is a Cartier divisor and hence can be pulled back to a Cartier divisor on X'. By the choices we made earlier, we have that $\varrho^*(mK_X + m\Delta) = mK_{X'} + m\Delta' + \Theta$ where Θ is an exceptional divisor.

However, using the fact that (X, Δ) is log canonical, one obtains that $\Theta \leq mG$. Combining this with (2) gives that

$$0 \leq \operatorname{div}_{X'}(g^m) + \varrho^*(mK_X + m\Delta) \leq m\big(\operatorname{div}_{X'}(g) + K_{X'} + \Delta' + G\big),$$

and in particular we obtain that

$$\mathrm{div}_{X'}(g) + K_{X'} + \Delta' + G \geq 0.$$

We claim that:

$$\mathrm{div}_{X'}(g) + K_{X'} + G \geq 0.$$

Proof. By construction

$$\mathrm{div}_{X'}(g) + K_{X'} + G = \varrho_*^{-1}(\underbrace{\mathrm{div}_X(g) + K_X}_{\geq 0}) + \underbrace{F + G}_{\text{exceptional}}. \tag{3}$$

Where F is an appropriate exceptional divisor, though it is not necessarily effective. We also have that

$$\mathrm{div}_{X'}(g) + K_{X'} + G = \underbrace{\mathrm{div}_{X'}(g) + K_{X'} + \Delta' + G}_{\geq 0} - \underbrace{D'}_{\text{non-exceptional}}. \tag{4}$$

Now let A be an arbitrary irreducible component of $\mathrm{div}_{X'}(g) + K_{X'} + G$. If A were not effective, it would have to be exceptional by (3) and non-exceptional by (4). Hence A must be effective and the claim is proven. □

It follows that $g \in \Gamma(X', \omega_{X'}(G)) = \Gamma(X, \varrho_* \omega_{X'}(G))$, completing the proof. □

15.1. **The log terminal and log canonical conditions for cones.** We study the condition that (Y, Δ_Y) has log canonical/terminal singularities when $Y = \operatorname{Spec} S$ is the affine cone over a projective variety X and Δ_Y corresponds to the pull-back of some $\mathbb{Q}$-divisor Δ_X on X via the k^*-bundle $Y \setminus V(S_+) \to X$ (or rather the closure of the pullback).

Suppose that (X, Δ_X) is a log $\mathbb{Q}$-Gorenstein pair and that A is an ample divisor. Set $S = \oplus H^0(X, \mathcal{O}_X(nA))$ to be the section ring and $Y = \operatorname{Spec} S$ and Δ_Y as above.

Proposition 15.14. *The pair (Y, Δ_Y) is klt (respectively lc) if and only if (X, Δ_X) is klt (respectively lc) and $-(K_X + \Delta_X) = rA$ for some $r \in \mathbb{Q}_{>0}$ (respectively $r \in \mathbb{Q}_{\geq 0}$).*

Remark 15.15. This proposition says that (X, Δ_X) is log Fano if and only if (Y, Δ_Y) is klt for some section ring. Likewise, (X, Δ) is log Calabi-Yau is equivalent to the condition that (Y, Δ_Y) is lc with lc-center at the origin.

Proof. Certainly the fact that (X, Δ_X) is klt/lc is necessary because of the k^*-bundle description of $Y \setminus V(S_+) \to X$ described above. For simplicity we assume now that A is (very (very)) ample. We can reduce to this case using Veronese cover tricks which I won't describe here.

First we ask ourselves what it means that $(K_Y + \Delta_Y)$ is $\mathbb{Q}$-Cartier (recall, that K_Y is just the sheaf associated to K_X via pull-back). This means that $n(K_Y + \Delta_Y)$ is locally free, and because we are working in the graded setting, this just means that $\mathcal{O}_Y(n(K_Y + \Delta_Y)) = \mathcal{O}_Y(m)$. But this is equivalent to the requirement that $n(K_X + \Delta_X) \sim mA$.

We now blow-up to origin of Y giving us a map $\pi : \widetilde{Y} \to Y$. There is one exceptional divisor E of this map and E is isomorphic to X. Furthermore, restricting $\mathcal{O}_{\widetilde{Y}}(-E)$ to E yields $\mathcal{O}_X(A)$.

Write $K_{\widetilde{Y}} - \pi^*(K_Y + \Delta_Y) = aE - \pi^{-1}_* \Delta_Y$. It is clear that $\pi^{-1}_*\Delta_Y|_E = \Delta_X$. However, we also know that $(K_{\widetilde{Y}} + E)|_E = K_X$. Rewriting our first equation gives us $\pi^*(K_Y + \Delta_Y) = K_{\widetilde{Y}} - aE + \pi^{-1}_*\Delta_Y$. Therefore

$$0 \sim (K_{\widetilde{Y}} + E - (a+1)E + \pi^{-1}_*\Delta_Y)|_E = K_X + (a+1)A + \Delta_Y$$

or in other words, $-(K_X + \Delta_Y) \sim (a+1)A$. In particular, if (Y, Δ) klt (respectively lc) then $a > 0$ (respectively $a \geq 0$). Thus $-(K_X + \Delta_Y)$ is some positive rational multiple of A (respectively, $-(K_X + \Delta_Y)$ is some non-negative multiple of A).

Conversely, if $-(K_X + \Delta_Y)$ is some positive rational multiple of A and (X, Δ_X) is klt, it can be shown that (Y, Δ_Y) is klt. We will not do this now though. There are two approaches, the most direct is to do a complete resolution of singularities followed by some analysis. The second is to use inversion of adjunction which allows one to relate the singularities of a divisor with the singularities of a pair. We'll cover more on this second topic later. $\square$

16. Pairs in positive characteristic

We've already studied pairs in a certain context. Consider pairs of the form (R, ϕ) where $\phi : F^e_* R \to R$ is an R-linear map. Our first goal will be to see that (R, ϕ) is very like a pair (X, Δ) where $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Proposition 16.1. *Suppose that X is a normal F-finite algebraic variety. Then there is a surjective map from non-zero elements $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ to $\mathbb{Q}$-divisors Δ such that $(p^e - 1)(K_X + \Delta) \sim 0$. Furthermore, two elements ϕ_1, ϕ_2 induce the same divisor if and only if there is a unit $u \in H^0(X, F^e_*\mathcal{O}_X)$ such that $\phi_1(u \cdot \_\_) = \phi_2(\_\_)$.*

More generally, there is a bijection of sets between effective $\mathbb{Q}$-divisors Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index[6] not divisible by $p > 0$ and certain equivalence relations on pairs $(\mathscr{L}, \phi : F^e_*\mathscr{L} \to \mathcal{O}_X)$ where $\mathscr{L}$ is a line bundle.

The equivalence relation described above is generated by equivalences of the following two forms.

- *Consider two pairs $(\mathscr{L}_1, \phi_1 : F^{e_1}\mathscr{L}_1 \to \mathcal{O}_X)$ and $(\mathscr{L}_2, \phi_2 : F^{e_2}\mathscr{L}_2 \to \mathcal{O}_X)$ where $e_1 = e_2 = e$. Then we declare these pairs equivalent if there is an isomorphism of line bundles $\psi : \mathscr{L}_1 \to \mathscr{L}_2$ and a commutative*

[6] The *index* of a $\mathbb{Q}$-Cartier divisor D is the smallest positive integer n such that $n(K_X + \Delta)$ is Cartier.

diagram:

$$\begin{tikzcd} F^e_* \mathscr{L}_1 \arrow[rr, "F^e_* \psi"] \arrow[dr, "\phi_1"'] & & \mathbb{F}^e_* \mathscr{L}_2 \arrow[dl, "\phi_2"] \\ & \mathcal{O}_X & \end{tikzcd}$$

- *Given a pair* $(\mathscr{L}, \phi : F^e_* \mathscr{L} \to \mathcal{O}_X)$*, we also declare it to be equivalent to the pair* $(\mathscr{L}^{p^{(n-1)e}+\cdots+1}, \phi^n : F^{ne} : \mathscr{L}^{p^{(n-1)e+\cdots+1}} \to \cdots \to \mathscr{L} \to \mathcal{O}_X)$*.*

First we do an example.

Example 16.2. Suppose R is a local ring and $X = \operatorname{Spec} R$. Further suppose that R is Gorenstein (or even such that $(p^e - 1)K_X$ is Cartier), then $\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R$ as we've seen. The generating map $\Phi_R \in \operatorname{Hom}_R(F^e_* R, R)$ corresponds to the zero divisor by the description above. Generally speaking, if $\psi(_) = \Phi_R(x \cdot _)$for $x \in F^e_* R$, then $\Delta_\psi = \frac{1}{p^e-1} \operatorname{div}_X(x)$. Even without the Gorenstein hypothesis, viewing $\operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta_\phi \rceil), R) \subseteq \operatorname{Hom}_R(F^e_* R, R)$, we have that ϕ generates $\operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta_\phi \rceil), R)$ as an $F^e_* R$-module.

Explicitly, consider $R = k[x]$. We know $\Phi_R : F^e_* R \to R$ is the map that sends x^{p^e-1} to 1 and the other relevant monomials to zero. Given a general element $\psi : F^e_* R \to R$ defined by the rule

$$\begin{aligned} x^{p^e-1} &\longmapsto a_0 \\ x^{p^e-2} &\longmapsto a_1 \\ \ldots &\longmapsto \ldots \\ x^1 &\longmapsto a_{p^e-2} \\ 1 &\longmapsto a_{p^e-1} \end{aligned}$$

Then $\psi(\_) = \Phi_R\left((a_0^{p^e} + a_1^{p^e} x + \cdots + a_{p^e-2} x^{p^e-2} + a_{p^e-1} x^{p^e-1}) \cdot \_\right)$ and so $\operatorname{div}_\psi = \frac{1}{p^e-1} \operatorname{div}(a_0^{p^e} + a_1^{p^e} x + \cdots + a_{p^e-2} x^{p^e-2} + a_{p^e-1} x^{p^e-1})$. One can do similarly easy computations for polynomial rings in general.

Now we give a proof of the proposition.

Proof. For the first equivalence, given $\phi \in \operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, F^e_* \mathcal{O}_X((1-p^e)K_X))$ define a divisor D_ϕ to be the effective divisor determined by ϕ linearly equivalent to $(1 - p^e)K_X$. Set $\Delta_\phi = \frac{1}{p^e-1} D_\phi$. It is easy to see that $(p^e - 1)(K_X + \Delta_\phi) \sim 0$.

Now, if ϕ_1 and ϕ_2 induce the same divisor, then $D_{\phi_1} = D_{\phi_2}$ which means that ϕ_1 and ϕ_2 are unit multiples of each other (as sections of $H^0(X, F^e_* \mathcal{O}_X((1 - p^e)K_X)))$ and the result follows.

For the more general statement, given $\phi : \operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathscr{L}, \mathcal{O}_X) \cong H^0(X, F^e_* \mathscr{L}^{-1}((1-p^e)K_X))$, we can associate a divisor D_ϕ such that $\mathcal{O}_X(D_\phi) \cong \mathscr{L}^{-1}((1-p^e)K_X)$

and define $\Delta_\phi = \frac{1}{p^e-1}D_\phi$. That the first equivalence relation holds is the same as in the case that $\mathscr{L} = \mathcal{O}_X$ above. The fact that the second equivalence relation holds, is an easy consequence of the following lemma. After the proof of this lemma, it is an easy exercise to verify that these two equivalence relations are all that is needed. $\square$

Before doing this lemma, let us do an example.

Lemma 16.3. *Suppose that $\mathscr{L}_1$ and $\mathscr{L}_2$ are line bundles and $\phi_1 : F^{e_1}\mathscr{L}_1 \to \mathcal{O}_X$ and $\phi_2 : F^{e_2}\mathscr{L}_2 \to \mathcal{O}_X$ are $\mathcal{O}_X$-linear maps. We can then define a composition of these maps as follows: Consider $\psi := \phi_2 \circ (F^{e_2}_*(\mathscr{L}_2 \otimes \phi_1)) : F^{e_1+e_2}\mathscr{L}_1 \otimes \mathscr{L}_2^{p^{e_1}} \to \mathscr{L}_2$. Then*

$$\Delta_\psi = \frac{p^{e_1}-1}{p^{e_1+e_2}-1}\Delta_{\phi_1} + \frac{p^{e_1}(p^{e_2}-1)}{p^{e_1+e_2}-1}\Delta_{\phi_2}$$

Notice that $\frac{p^{e_1}-1}{p^{e_1+e_2}-1} + \frac{p^{e_1}(p^{e_2}-1)}{p^{e_1+e_2}-1} = 1$.

Proof. The statement is local, so we may assume that $\mathscr{L}_1 \cong \mathscr{L}_2 \cong \mathcal{O}_X$. In fact, we may assume that X is the prime spectrum of a DVR R with parameter r. Fix $\Psi_R : F_*R \to R$ to be the generating map of $\operatorname{Hom}_R(F_*R, R)$ as an F_*R-module.

In this case, $\phi_1(\_\_) = \Psi^{e_1}_R(x_1 \cdot \_\_)$ and $\phi_2(\_\_) = \Psi^{e_2}_R(x_2 \cdot \_\_)$ where $x_i \in F^{e_i}_* R$ and so $\Delta_{\phi_i} = \frac{1}{p^{e_i}-1}\operatorname{div}_X(x_i)$. Then

$$\phi_2(F^{e_2}_*\phi_1(\_\_)) = \Psi^{e_2}_R\left(F^{e_2}_* x_2 \Psi^{e_1}_R\left(F^{e_1}_* x_1 \_\_\right)\right) = \Psi^{e_1+e_2}\left(F^{e_1+e_2}_* x_1 x_2^{p^{e_1}}\_\_\right)$$

The divisor of this composition is evidently

$$\begin{aligned} & \tfrac{1}{p^{e_1+e_2}-1}(\operatorname{div}(x_1) + p^{e_1}\operatorname{div}(x_2)) \\ = & \tfrac{1}{p^{e_1+e_2}-1}(\operatorname{div}(x_1) + p^{e_1}\operatorname{div}(x_2)) \\ = & \tfrac{p^{e_1}-1}{p^{e_1+e_2}-1}\Delta_{\phi_1} + \tfrac{p^{e_1}(p^{e_2}-1)}{p^{e_1+e_2}-1}\Delta_{\phi_2} \end{aligned}$$

$\square$

Lemma 16.4. *An element $\phi \in \operatorname{Hom}_R(F^e_*R, R)$ is contained inside the submodule*

$$\operatorname{Hom}_R(F^e_*R(\lceil (p^e-1)\Delta\rceil), R) \subseteq \operatorname{Hom}_R(F^e_*R, R) \tag{5}$$

if and only if $D_\phi \geq (p^e-1)\Delta$.

Proof. Because all the module are reflexive the statement can be reduced to the case when R is a discrete valuation ring and $\Delta = s\operatorname{div}(x)$ where x is the parameter for the DVR R and $s \geq 0$ is a real number. In this case, the inclusion from equation 5 can be identified with the multiplication map $R \to R$ which sends 1 to $x^{\lceil s(p^e-1)\rceil}$. Thus, $\phi \in \operatorname{Hom}_R(F^e_*R, R) \cong R$ is contained inside $\operatorname{Hom}_R(F^e_*R(\lceil (p^e-1)\Delta\rceil), R) \cong x^{\lceil s(p^e-1)\rceil}R$ if and only if $D_\phi \geq \lceil s(p^e-1)\rceil\operatorname{div}(x) = \lceil (p^e-1)\Delta\rceil$. However, since D_ϕ is integral, it is harmless to remove the round-up $\lceil\cdot\rceil$. $\square$

Remark 16.5. One can work with non-effective divisors similarly. One then can consider maps $\phi : F^e_* \mathscr{L} \to K(X)$ where $K(X)$ is the fraction field of X.

Definition 16.6. [HH89], [HW02], [Tak04a], [Sch08b] Suppose that $(X, \Delta, \mathfrak{a}^t)$ is a triple where X is an F-finite normal scheme, Δ is an effective $\mathbb{Q}$-divisor, $\mathfrak{a}$ is an ideal sheaf and $t \geq 0$ is a real number. Further suppose that X is the spectrum of a local ring R. We say that $(X, \Delta, \mathfrak{a}^t)$ is:

(a) *sharply F-pure* if there exists some $e > 0$ and some $\phi \in \mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R)$ such that $1 \in \phi(F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil})$.
(b) *strongly F-regular* if for every $c \in R \setminus 0$, there exists a $e > 0$ and some $\phi \in \mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R)$ such that $1 \in \phi(F^e_* c\mathfrak{a}^{\lceil t(p^e-1) \rceil})$.

If X is not the spectrum of a local ring, then we generalize these definitions by requiring them at every point. They are open conditions.

Suppose that X is quasi-projective. The *(big) test ideal of* $(X, \Delta, \mathfrak{a}^t)$, denoted $\tau_b(X, \Delta, \mathfrak{a}^t)$ is defined to be the unique smallest non-zero ideal of $J \subseteq R$ such that $\phi(F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil} J \mathscr{L}) \subseteq J$ for every $\phi : F^e_* \mathscr{L} \to \mathcal{O}_X$ such that $\Delta_\phi \geq \Delta$. This always exists and its formation commutes with localization.

Definition 16.7. We say that R is *strongly F-regular / F-pure* if the same statement holds for $\Delta = 0$ and $\mathfrak{a} = R$.

Remark 16.8. If $\Delta = \Delta_\psi$ for some $\psi : F^e_* \mathscr{L} \to \mathcal{O}_X$, then in the definition of the big test ideal / sharp F-purity / strong F-regularity, one only needs to check the condition for $\phi = \psi^n$.

Proposition 16.9. *A ring is strongly F-regular if and only if $\tau_b(R) = R$. Furthermore a strongly F-regular ring is always F-rational (in particular, it is Cohen-Macaulay) and a Gorenstein F-rational ring is strongly F-regular.*

Proof. Suppose R is strongly F-regular and local and suppose that J satisfies $\phi(F^e_* J) \subseteq J$ for every $\phi : F^e_* R \to R$. The strong F-regularity hypothesis implies immediately that J contains R and is thus equal to 1. Conversely, suppose that $\tau_b(R) \neq R$, then choose any element $0 \neq c \in \tau_b(R)$. It follows that for every $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, $\phi(F^e_* cR) \subseteq \tau_b(R)$ which does not contain 1.

We've already seen that F-rational Gorenstein rings are strongly F-regular. This is simply because $R = \omega_R$ and in this case we have a map $\Psi_R : F^e_*(\omega_R = R) \to (\omega_R = R)$ such that $\tau(R, \Psi_R) = R$ (interestingly, we don't need the Cohen-Macaulay condition here, it is implied for free by what follows).

Now assume that R is strongly F-regular, we will show it is F-rational and in particular Cohen-Macaulay (this is one proof where I think it is easier to use the tight closure definitions). First note that R is necessarily normal since we know the conductor is ϕ-compatible for all $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, so in particular $\tau_b(R)$ is always contained in the conductor. We now show that R is Cohen-Macaulay by showing that $h^i(\omega_R^{\bullet}) = 0$ for all $i > -\dim R$. Suppose not, so choose $0 \neq c \in R$ such that $ch^i(\omega_R^{\bullet}) = 0$ but $h^i(\omega_R^{\bullet}) \neq 0$ (such modules always have support strictly smaller in dimension than the ring so this is possible). Dual to the map $R \to F^e_* R$ which sends $1 \mapsto c$, we have the map

$$h^i(F^e_* \omega_R^\bullet) \xrightarrow{F^e_* \times c} h^i(F^e_* \omega_R^\bullet) \longrightarrow h^i(\omega_R^\bullet)$$

For e large enough, this map is necessarily surjective (since our map $R \to F^e_* R$ splits), but this is ridiculous since it is also zero.

Using the same argument, we also have that the composition $F^e_* c\omega_R \subseteq F^e_* \omega_R \to \omega_R$ surjects for all $e \gg 0$. But this clearly implies that $\tau(\omega_R) = \omega_R$ which completes the proof. $\square$

Remark 16.10. We have the following implications:

$$\begin{tikzcd} \text{strongly } F\text{-regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-rational} \arrow[d, Rightarrow] \\ F\text{-pure} \arrow[r, Rightarrow] & F\text{-injective} \end{tikzcd}$$

Furthermore, under the (quasi)-Gorenstein hypothesis the horizontal arrows can be reversed.

Proposition 16.11. *The ideal $\tau_b(X, \Delta, \mathfrak{a}^t)$ exists.*

Proof. I'll only prove this for $X = \operatorname{Spec} R$. Choose a non-zero $\psi \in M^e_{\Delta,\mathfrak{a}^t} = (F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil}) \cdot \operatorname{Hom}_R(F^e_* R(\lceil (p^e-1)\Delta \rceil), R)$. We view ψ as a map from $F^e_* R$ to R. Choose c a test element for the pair (R, ψ). Then we claim that

$$\tau_b(R, \Delta, \mathfrak{a}^t) = \sum_{e \geq 0} \sum_{\phi \in M^e_{\Delta,\mathfrak{a}^t}} \phi(F^e_* cR).$$

It is enough to show equality after localizing at each prime ideal, and so we may assume R is local. The sum is stabilized by all $\phi \in M^e_{\Delta,\mathfrak{a}^t}$. There is a computation here to check this, that the elements of $M^e_{\Delta,\mathfrak{a}^t}$ form an algebra of maps, but it is of the form $p^d \lceil (p^e-1)t \rceil + \lceil (p^d-1)t \rceil \geq \lceil p^{e+d} - 1 \rceil$. On the other hand, clearly any $J \subseteq R$ that is stabilized by all $\phi \in M^e_{\Delta,\mathfrak{a}^t}$ contains c since all powers of ψ live in $M^e_{\Delta,\mathfrak{a}^t}$ for various e. $\square$

17. F-singularities and birational maps

Our goal in this section is to relate F-singularities and test ideals with log canonical and log terminal singularities as well as multiplier ideals. In order to do this, we need to explain how maps $\phi : F^e_* R \to R$ behave under birational maps.

Proposition 17.1. *Suppose that $\pi : \widetilde{X} \to X$ is a proper birational map and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. Write*

$$K_{\widetilde{X}} - \sum a_i E_i = f^*(K_R + \Delta_\phi)$$

Then ϕ induces a map $\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)\sum a_i E_i) \to \mathcal{O}_{\widetilde{X}}$ which agrees with ϕ where π is an isomorphism. Finally, it induces a map (which we also call $\widetilde{\phi}$)

$$\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil).$$

Proof. Throughout, we remove the singular locus of $\widetilde{X}$ if necessary so that it is regular, and work with divisors on this locus. This is harmless though since we are looking at maps between reflexive modules.

By assumption ϕ generates $\mathrm{Hom}_R(F^e_* R(\lceil (p^e-1)\Delta_\phi \rceil), R) \cong F^e_* R((1-p^e)(K_R + \Delta_\phi)) \cong F^e_* R$. Thus we have a section $d \in f^* R((1-p^e)(K_R+\Delta_\phi)) \cong \mathcal{O}_{\widetilde{X}}$ corresponding to ϕ, and furthermore this section generates. So that we obtain a section $d \in \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}((1-p^e)(K_{\widetilde{X}} - \sum a_i E_i))$ which generates as an $\mathcal{O}_{\widetilde{X}}$-module. However, $F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)(K_{\widetilde{X}} - \sum a_i E_i)) = \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)(\sum a_i E_i)), \mathcal{O}_{\widetilde{X}})$ and we obtain our first statement easily.

For the second statement, consider $\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)\sum a_i E_i) \to \mathcal{O}_{\widetilde{X}}$. Twisting by $\mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil)$ gives us a map

$$\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)\sum a_i E_i + p^e \lceil \sum a_i E_i \rceil) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil)$$

However, $(1-p^e)\sum a_i E_i + p^e \lceil \sum a_i E_i \rceil \geq \lceil (1-p^e)\sum a_i E_i + p^e \sum a_i E_i \rceil = \lceil \sum a_i E_i \rceil$ which gives the desired map via composition with the inclusion. $\square$

Remark 17.2. Restrict the above map $\widetilde{\phi}$ to an E_i such that $a_i \leq 0$. Localizing at the generic point of that E_i gives us a "generating" map from $\mathcal{O}_{\widetilde{X}, E_i}((1-p^e)a_i E_i) \to \mathcal{O}_{\widetilde{X}, E_i}$. In other words, if we pay close attention to our embedding into the fraction field, the divisor associated to $\widetilde{\phi}$ corresponds to $\sum -a_i E_i$ (at least for those E_i with non-positive a_i). As we've previously alluded to, one can work with anti-effective divisors too, in that case $\widetilde{\phi}$ corresponds to $-\sum a_i E_i$.

Remark 17.3. In fact, for any effective divisor E on $\widetilde{X}$, $\pi_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil + E)$ is also stabilized by ϕ.

Remark 17.4. This immediately implies the inclusion $\tau_b(R, \Delta_\phi) \subseteq \mathcal{J}(R, \Delta_\phi)$ assuming the existence of resolutions of singularities in characteristic $p > 0$. In fact, a slight modification of this implies that $\tau_b(R, \Delta, \mathfrak{a}^t) \subseteq \mathcal{J}(R, \Delta, \mathfrak{a}^t)$ under the assumption that $K_X + \Delta$ is $\mathbb{Q}$-Gorenstein. To see this, assume that R is local notice that for every $\psi \in M^e_{\Delta, \mathfrak{a}^t}$, we have that $\Delta_\psi = \Delta_{\psi'} + \frac{1}{p^e - 1}\operatorname{div}(f)$ where $\Delta_{\psi'} \geq \Delta$ and $f \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$. It easily follows from the method of the proof and Remark 17.3 above that $\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG \rceil)$ is ψ-stable.

We'd now like to relate F-pure and log canonical singularities.

Theorem 17.5. [HW02] *Suppose that $(X, \Delta, \mathfrak{a}^t)$ has F-pure singularities and that $K_X + \Delta$ is $\mathbb{Q}$-Gorenstein. Further suppose that $\pi : \widetilde{X} \to X$ is a proper* *birational map with $\widetilde{X}$ normal and $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-G)$. Then if we write*

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG = \sum a_i E_i$$

we have that each $a_i \geq -1$.

Proof. Without loss of generality, we may assume that X is the spectrum of a local ring. We choose $\psi \in M^e_{\Delta, \mathfrak{a}^t}$ which induces a surjective map $\psi : F^e_* R \to R$. We notice that if we write

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta_\psi) = \sum b_i E_i$$

then all of the $b_i \leq a_i$ and so it suffices to prove the statement for the b_i.

Suppose then that one of the $b_i < -1$. Localize at the generic point of the associated E_i. This gives us a DVR $\mathcal{O}_{\widetilde{X}, E_i}$ and a map $\widetilde{\psi} : F^e_* \mathcal{O}_{\widetilde{X}, E_i} \to \mathcal{O}_{\widetilde{X}, E_i}$ that is also surjective. Furthermore, the divisor corresponding to $\widetilde{\psi}$ is $-b_i E_i$. Therefore, our result follows from the following lemma:

Lemma 17.6. *If (S, Δ) is F-pure with Δ effective, then $\lceil \Delta \rceil$ is reduced (in other words, the coefficients of Δ are less than or equal to 1).*

Proof. Without loss of generality we may assume that S is a DVR with parameter s. Write $\Delta = \lambda \operatorname{div}(s)$. Suppose that $\lambda > 1$, we will show that (S, Δ) is not F-pure. Let Ψ_S be the generating map of $\operatorname{Hom}_S(F^e_* S, S)$. Then for any $\phi \in M^e_\Delta$, we have $\phi(\_) = \Phi_S(x \cdot \_)$ where $x = us^m$ and $m \geq \lceil (p^e - 1)\lambda \rceil \geq p^e$. But then clearly $\phi(z) \subseteq (s)$ for all $z \in F^e_* S$ proving that no ϕ can be surjective. ☐

☐

Corollary 17.7. [MvdK92] *Suppose that X is a normal variety and $\pi : \widetilde{X} \to X$ is a projective birational map with normal $\widetilde{X}$. If there exists a map $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ such that*

(a) *$(X, \Delta_\phi) = (X, \phi)$ is strongly F-regular.*
(b) *If we write $K_{\widetilde{X}} - \pi^*(K_X + \Delta) = \sum a_i E_i$ then all a_i satisfy $-1 < a_i \leq 0$ (note the lower bound follows from (a)).*

Then $R^i \pi_* \omega_{\widetilde{X}} = 0$ for all $i > 0$. In fact, $R^i \pi_* h^j(\omega^{\bullet}_{\widetilde{X}}) = 0$ for all j.

Proof. The statement is local so we may assume that X is the spectrum of a local ring R. Fix an anti-effective relatively π-ample Weil divisor E on $\widetilde{X}$ and choose an element $d \in R$ such that $\operatorname{div}_{\widetilde{X}}(d) \geq -E$. By the first hypothesis, there exists an $n \gg 0$ such that $\phi^n(F^{ne}_* dR) = R$ say $\phi^n(F^{ne}_* dc) = 1$. Consider the map $\psi : F^{ne}_* R \to R$ defined by $\phi(\_) = \phi(cd \cdot \_)$, noting that $\Delta_\psi \geq \Delta_\phi$. Write $K_{\widetilde{X}} - \pi^*(K_X + \Delta) = \sum b_i E_i$ and observe that $-1 \leq b_i < 0$ (actually, $b_i = a_i - \frac{1}{p^{ne} - 1} \operatorname{div}_{E_i}(cd)$). We also induce a map $\widetilde{\psi} : F^e_* \mathcal{O}_{\widetilde{X}}((1 - p^{ne}) \sum b_i E_i) \to \mathcal{O}_{\widetilde{X}}$ which sends 1 to 1. All of the a_i and b_i are non-positive, and so we have an inclusion $\mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}((1-p^{ne})\sum b_i E_i)$. In fact, by construction we have that

$$\mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}(-E) \subseteq \mathcal{O}_{\widetilde{X}}(\mathrm{div}_{\widetilde{X}}(d)) \subseteq \mathcal{O}_{\widetilde{X}}((1-p^{ne})\sum b_i E_i).$$

In particular, $\mathcal{O}_{\widetilde{X}}$ is Frobenius split, and we can express the splitting as the isomorphism

$$\mathcal{O}_{\widetilde{X}} \to F^{ne}_* \mathcal{O}_{\widetilde{X}} \to F^{ne}_* \mathcal{O}_{\widetilde{X}}(-E) \to \mathcal{O}_{\widetilde{X}}.$$

Iterating this isomorphism m-times, we obtain the isomorphism

$$\mathcal{O}_{\widetilde{X}} \to F^{mne}_* \mathcal{O}_{\widetilde{X}} \to F^{mne}_* \mathcal{O}_{\widetilde{X}}(-(1+p+\cdots+p^{m-1})E) \to \mathcal{O}_{\widetilde{X}}$$

The idea will be we can use Frobenius to amplify the amplitude of E.

Dualizing, we obtain that

$$\omega^{\bullet}_{\widetilde{X}} \leftarrow F^{mne}_* \omega^{\bullet}_{\widetilde{X}} \leftarrow F^{mne}_* \omega^{\bullet}_{\widetilde{X}}((1+p+\cdots+p^{m-1})E) \leftarrow \omega^{\bullet}_{\widetilde{X}}$$

also an isomorphism. Taking cohomology gives us an isomorphism

$$h^j(\omega^{\bullet}_{\widetilde{X}}) \leftarrow F^{mne}_* h^j(\omega^{\bullet}_{\widetilde{X}}) \leftarrow F^{mne}_* h^j(\omega^{\bullet}_{\widetilde{X}})((1+p+\cdots+p^{m-1})E) \leftarrow h^j(\omega^{\bullet}_{\widetilde{X}}).$$

Applying $R^i\pi_*$ gives us the desired conclusion since E is anti-ample and we may take $m \gg 0$. $\square$

We now relate the multiplier ideal and the test ideal.

Theorem 17.8. [Smi00b], [Har05], [HY03], [Tak04b] *Suppose that* $(X_0 = \operatorname{Spec} R_0, \Delta_0, \mathfrak{a}_0^t)$ *is a triple in characteristic zero such that* $K_{X_0} + \Delta_0$ *is* $\mathbb{Q}$-*Cartier. Then for all* $p \gg 0$, $(\mathcal{J}(X, \Delta, \mathfrak{a}^t))_p = \tau(X_p, \Delta_p, \mathfrak{a}_p^t)$.

Proof. We will be doing reduction to characteristic $p > 0$ here. We will not write the subscript p (although will write the subscript 0). We first recall Hara's lemma on surjectivity of the dual Frobenius map (which we still haven't proved).

Lemma 17.9. [Har98] *Suppose that* R_0 *is a ring of characteristic zero,* $\pi : \widetilde{X}_0 \to \operatorname{Spec} R_0$ *is a log resolution of singularities,* D_0 *is a* π-*ample* $\mathbb{Q}$-*divisor with simple normal crossings support. We reduce this setup to characteristic* $p \gg 0$. *Then the natural map*

$$(F^e)^{\vee} = \Phi_{\widetilde{X}} : \pi_* F^e_* \omega_{\widetilde{X}}(\lceil p^e D \rceil) \to \pi_* \omega_{\widetilde{X}_p}(\lceil D \rceil)$$

surjects.

Fixing a log resolution $\widetilde{X}_0$ of X_0 we write $\mathfrak{a}_0 \cdot \mathcal{O}_{\widetilde{X}_0} = \mathcal{O}_{\widetilde{X}_0}(-G_0)$ and reduce this setup to characteristic $p > 0$. We choose $c_0 \in \mathcal{O}_{X_0}$ an element whose power is going to be a test element in characteristic $p \gg 0$, and then further multiply it by the product of the generators of the a_i. We choose a relatively ample divisor exceptional E_0 in characteristic zero such that $\lceil -\pi^*(K_{X_0}+\Delta_0) - tG_0 + E_0 - \varepsilon \operatorname{div}_{\widetilde{X}_0}(c_0) \rceil = \lceil -\pi^*(K_{X_0}+\Delta_0) - tG_0 + E_0 \rceil$ and also reduce it to characteristic $p > 0$. Our D_0 is going to be $E_0 - \pi^(K_{X_0}+\Delta_0) - tG_0 - \varepsilon \operatorname{div}_{\widetilde{X}_0}(c_0)$.

After reduction to characteristic $p \gg 0$, we may assume that $K_X + \Delta_X$ is $\mathbb{Q}$-Cartier with index not divisible by p. Therefore, we may choose a $\phi : F^e_* R \to R$ corresponding to Δ_X as before. As we've noted, this induces a map

$$\widetilde{\phi} : F^e_* \omega_{\widetilde{X}}(\lceil -\pi^*(K_X+\Delta) - tp^e G + p^e E + p^e \varepsilon \operatorname{div}_{\widetilde{X}}(c) \rceil) \to \omega_{\widetilde{X}}(\lceil -\pi^*(K_X+\Delta) - tG + E + \varepsilon \operatorname{div}_{\widetilde{X}}(c) \rceil)$$

We claim that this map can be identified with:

$$\begin{array}{rcl} (F^e)^\vee & : & F^e_* \omega_{\widetilde{X}}(\lceil -p^e \pi^*(K_X + \Delta) - tp^e G + p^e E + p^e \varepsilon \operatorname{div}_{\widetilde{X}}(c) \rceil) \\ & \to & \omega_{\widetilde{X}}(\lceil -\pi^*(K_X + \Delta) - tG + E + \varepsilon \operatorname{div}_{\widetilde{X}}(c) \rceil) \end{array}$$

Given this claim, $\widetilde{\phi}$ surjects. Now argue as we did for rational singularities. For $e \gg 0$, π_* of the domain of $\widetilde{\phi}$ is contained inside

$$F^e_* c^n \overline{\mathfrak{a}^{\lceil t(p^e-1) \rceil}}$$

where c^{n-1} is a test element. The problem is the integral closure. We need $c\overline{\mathfrak{a}^{\lceil t(p^e-1) \rceil}} \subseteq \mathfrak{a}^{\lceil t(p^e-1) \rceil}$. But c factors as both a test element d of R as well as the product of generators of $\mathfrak{a}$. Therefore, $c\overline{\mathfrak{a}^{\lceil t(p^e-1) \rceil}} \subseteq d\overline{\mathfrak{a}^{\lceil t(p^e-1) \rceil + r}}$ where r is the number of generators of R. The tight-closure Briançon-Skoda theorem (which we may prove a little later, [HH90]) tells us that this is contained in $\mathfrak{a}^{\lceil t(p^e-1) \rceil}$ as desired. Then the sum of images of these maps (for $e \gg 0$) is the test ideal.

To prove the claim, we argue as follows. Notice first that $(F^e)^\vee : F^e_* \mathcal{O}_{\widetilde{X}}((1 - p^e)K_{\widetilde{X}}) \to \mathcal{O}_{\widetilde{X}}$ is (locally) the generating map as is $\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}((p^e - 1)\pi^*(K_X + \Delta) - (p^e - 1)K_{\widetilde{X}}) \to \mathcal{O}_{\widetilde{X}}$. But $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + (p^e - 1)\pi^*(K_X + \Delta)) \cong F^e_* \mathcal{O}_{\widetilde{X}}((1 - p^e)K_{\widetilde{X}})$ so the two maps are actually the same (up to multiplication by a unit). From there, the more complicated maps above were then obtained by twisting by the same $\mathbb{Q}$-divisors, and then doing the same inclusions. $\square$

Corollary 17.10. *A triple $(X, \Delta, \mathfrak{a}^t)$ in characteristic zero is Kawamata log terminal if and only if it is of open strongly F-regular type.*

Remark 17.11. The following diagram explains the singularities we understand and the implications between them.

$$\begin{tikzcd}
\text{Canonical} \arrow[d, Rightarrow] & & & \\
\text{Log Terminal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[rr, bend left, Leftrightarrow] & \text{Rational} \arrow[d, Rightarrow] \arrow[rr, bend left, Leftrightarrow] & F\text{-Regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-Rational} \arrow[d, Rightarrow] \\
\text{Log Canonical} \arrow[r, Rightarrow] & \text{Du Bois} \arrow[l, bend left=60, Rightarrow, "+\text{ Gor. \& normal}"] & F\text{-Pure}/F\text{-Split} \arrow[r, Rightarrow] \arrow[ll, bend left, Rightarrow] & F\text{-Injective} \arrow[l, bend left=60, Rightarrow, "+\text{ Gor.}"] \arrow[ll, bend left, dotted, Rightarrow]
\end{tikzcd}$$

It is an open question whether Du Bois singularities have dense F-injective type or whether log canonical singularities have dense F-pure type.

18. Characteristic $p > 0$ analogs of LC-centers and subadjunction

We recall the following definition (for now, we work in characteristic zero). Most of the results found here (including more details of various proofs) can be found in [Sch09].

Definition 18.1. Suppose (X, Δ) is a pair and $W \subseteq X$ is an irreducible subvariety set η to be the generic point of W. We say that W is a *non-KLT-center* if there exists a divisor E_i on some birational model $\pi : \widetilde{X} \to X$ of X such that $W = \pi(E_i)$ where the discrepancy $a_i \leq -1$ (as usual, $\sum a_i E_i = K_{\widetilde{X}} - \pi^*(K_X + \Delta)$. We say that W is an *LC-center* if W is a non-KLT-center and furthermore, $(X, \Delta)_\eta$ is log canonical.

Lemma 18.2. *Given a pair (X, Δ) as above further assume that X is affine. Then $W \subseteq X$ is a non-KLT-center if and only if for every $d \in \mathcal{O}_X$ such that $\eta \in V(d)$, we have that for every $\varepsilon > 0$ that the pair $(X, \Delta + \varepsilon \operatorname{div}_X(d))_\eta$ is not log canonical.*

Proof. If W is a non-KLT-center, then the conclusion of the lemma is obvious. Suppose conversely that W satisfies the condition of the lemma but is not an LC-center.

If η is a codimension 1 point, then the result is also clear (no birational models are needed). On the other hand, if $(X, \Delta)_\eta$ is not log canonical, we are already done, so we may assume that $(X, \Delta)_\eta$ is log canonical. Choose a log resolution $\pi : \widetilde{X} \to X$ of (X, Δ) such that $I_W \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$ is also a SNC divisor (and by hypothesis, all the discrepancies of E_i such that $\pi(E_i) = W$ satisfy $a_i > -1$). We choose a general element d of $\mathcal{O}_X$ vanishing at η so that π is a log resolution of $(X, \Delta + \varepsilon \operatorname{div}_X(d))$ for every $\varepsilon > 0$. Because d is general, for $0 < \varepsilon \ll 1$ $(X, \Delta + \varepsilon \operatorname{div}_X(d))$ is log canonical on $X \setminus W$. Of course, for E_i such that $\pi(E_i) = W$, the associated a_i for $(X, \Delta + \varepsilon \operatorname{div}_X(d))$ is still > -1 for $\varepsilon > 0$ small enough. But this implies that $(X, \Delta + \varepsilon \operatorname{div}_X(d))_\eta$ is log canonical. $\square$

In analogy with the previous lemma, we make the following definition.

Definition 18.3. Suppose that $X = \operatorname{Spec} R$ is an F-finite normal scheme of characteristic $p > 0$ and that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $(1 - p^e)(K_X + \Delta)$ is Cartier. For an element $Q \in \operatorname{Spec} R$, we say that $V(Q) = W \subseteq X$ is an *non-F-regular-center* if for every element $d \in Q$ and every $\varepsilon > 0$, we have that $(X, \Delta + \varepsilon \operatorname{div}(d))_Q$ is not F-pure. It is an *F-pure center* if we additionally require that $(X, \Delta)_Q$ is F-pure. These definitions generalize to the non-affine setting by requiring them on affine charts.

Lemma 18.4. *With notation as above, suppose that ϕ to be a map $F^e_* \mathscr{L} \to \mathcal{O}_X$ a map corresponding to Δ. Then W is a non-F-regular center if and only if $\phi(F^e_* Q \mathscr{L}) \subseteq Q$.*

Proof. Without loss of generality, we may assume that R is a local ring and thus that $\mathscr{L} = \mathcal{O}_X$. Furthermore, we can localize at Q and assume that Q is the maximal ideal of R. First we claim that $\phi(F^e_* Q) \subseteq Q$ if and only if $\phi^n(F^{ne}_* Q) \subseteq Q$ for some $n > 0$. The ($\Rightarrow$) direction is clear, for the reverse, if $\phi(F^e_* Q) \nsubseteq Q$, then $\phi(F^e_* Q) = R$, but then it follows easily that $\phi^n(F^{ne}_* Q) = R$ for all $n > 0$. From this, it follows that $\phi(F^e_* Q) \subseteq Q$ if and only if for every $\psi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ such that $\Delta_\psi \geq \Delta_\phi$ we have that $\psi(F^e_* Q) \subseteq Q$.

Of course, we may assume that the $\varepsilon > 0$ we consider is of the form $\frac{1}{p^{ne}-1}$. Now, $(X, \Delta + \frac{1}{p^{ne}-1} \operatorname{div}(d))$ is not F-pure if and only if $\phi^n(F^{ne}_* dR) \subseteq Q$. But if we require this for all $d \in Q$, this just says that $\phi^n(F^{ne}_* Q) \subseteq Q$. $\square$

Corollary 18.5. *(X, Δ) is strongly F-regular if and only if it has no non-F-regular centers.*

Theorem 18.6. *Non-klt centers in characteristic zero reduce to non-F-regular centers in characteristic $p > 0$.*

Proof. It follows easily from the fact that we have the map $\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil)$, which induces for each effective divisor G,

$$\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil + G) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil + G).$$

The theorem then follows once one observes that any non-klt center can be written as $\pi_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil + G)$ for some appropriate G. $\square$

Here is a characteristic $p > 0$ version of Kawamata's subadjunction theorem.

Theorem 18.7. *Suppose that (X, Δ) is a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by $p > 0$. Suppose that $W \subseteq X$ is a normal F-pure center. Then, there exists a canonically determined divisor Δ_W on W such that $K_W + \Delta_W \sim_{\mathbb{Q}} (K_X + \Delta)|_W$ and such that:*

- *(W, Δ_W) is F-pure if and only if (X, Δ) is F-pure near W.*
- *(W, Δ_W) is strongly F-regular if and only if W is minimal with respect to inclusion of F-pure centers with respect to containment.*
- *The set of F-pure centers of (W, Δ_W) is the same as the set of F-pure centers of (X, Δ) which properly contain W.*

Proof. Given a map $\phi : F^e_* \mathscr{L} \to \mathcal{O}_X$ corresponding to Δ, suppose that Q is an ideal sheaf such that $V(Q) = W$. We immediately obtain a map $\phi|_W : F^e_* \mathscr{L}|_W \to \mathcal{O}_W$ obtained by modding out by Q. This map $\phi|_W$ corresponds to divisor Δ_W. The first statement follows from the fact that in a local ring, $\phi : F^e_* R \to R$ surjects if and only if the induced map $\phi : F^e_* R/Q \to R/Q$ surjects. The third statement follows the fact that $P \supseteq Q$ is ϕ-compatible if and only if P/Q is ϕ/Q-compatible, and the third statement implies the second. $\square$

Remark 18.8. If W is not normal, one can always induce a divisor Δ_{W^N} on the normalization of W. Nice properties of (X, Δ) still induce nice properties of (W^N, Δ_{W^N}) but the converse statements don't necessarily hold (this seems to be due to inseparability and wild ramification in the normalization map $\eta : W^N \to W$).

If one knew that log canonical implied F-pure type, one could prove a number of interesting things about log canonical centers via reduction to characteristic $p > 0$.

This is very different from the behavior in characteristic zero. In particular, Δ_W is canonically determined which is not the case in characteristic zero. Consider the following example.

Example 18.9 (Speyer, –, Xu)**.** Suppose that $X \to \operatorname{Spec} k[t] = \mathbb{A}^1$ is the family of cones over elliptic curves defined by $zy^2 - x^3 + txz^2$ with a section $\sigma : \mathbb{A}^1 \to X$ mapping to the cone points. Further assume that there is a log resolution $\pi : \widetilde{X} \to X$ which is obtained by blowing up the image of σ (which we now call Z). Finally note that X is F-pure at the generic point of Z.

It then follows that Z is an F-pure center. Note that X is $\mathbb{Q}$-Gorenstein with index not divisible by $p > 0$, so we can set $\Delta = 0$. We can construct Δ_W as above. In this context, Δ_W has support exactly at those points such that the associated elliptic curve is not F-split (ie, supersingular).

Note that $(X, \operatorname{div}_X(t-\lambda))$ has a log canonical center at $W = (x, y, z, t-\lambda)$. Furthermore, by blowing up $\widetilde{X}$ at the inverse image of that point, one obtains a log resolution with two exceptional divisors, the one dominating Z and the one dominating W. Both of these exceptional divisors have discrepancy -1. It then follows that if $(X, \operatorname{div}_X(t-\lambda))$ is F-pure, the exceptional divisor is F-split. This implies that the associated elliptic curve is also F-split. But $(X, \operatorname{div}_X(t-\lambda))$ is F-pure if and only if $(W, \Delta_W + \operatorname{div}_W(t-\lambda))$ is F-pure. The latter is F-pure at W if and only if Δ_W does not have $\operatorname{div}_W(t-\lambda)$ as a component. This implies that if λ corresponds to a supersingular elliptic curve, then Δ_W must have $\operatorname{div}_W(t-\lambda)$ among its components.

Conversely, suppose that λ corresponds to an ordinary elliptic curve E_λ. The generating map on the associated elliptic curve $\psi : F_*\mathcal{O}_{E_\lambda} \to \mathcal{O}_{E_\lambda}$ is always the map induced by the pair $(X, \operatorname{div}_X(t-\lambda))$ on $\widetilde{X}$ as above. On the elliptic curve, the map ψ sends units to units, thus on $\widetilde{X}$, the map associated to $(X, \operatorname{div}_X(t-\lambda))$ has to send units to non-zero elements which restrict to units on E_λ. Thus back on X, units must be sent to elements that are units near W and the proof is complete.

We give one more application of these ideas. I do not know of an analog of this result in characteristic zero.

Theorem 18.10. *Suppose that S is a regular local ring and that $R = S/I$ is any reduced normal ring and Δ_R is a divisor on* $\operatorname{Spec} R$ *such that $K_R + \Delta_R$ is F-pure with index not divisible by $p > 0$. Then there exists a divisor Δ_S on S such that $K_S + \Delta_S$ is* $\operatorname{Spec} R \subseteq \operatorname{Spec} S$ *is an F-pure center of (S, Δ_S) and* *furthermore, that Δ_S and Δ_R are related as in Theorem 18.7. In particular, (R, Δ_R) is F-pure, then we may choose (S, Δ_S) also to be F-pure.*

Using the same idea (Fedder's lemma), we have the following method for checking whether an ideal is a non-F-regular center.

Proposition 18.11. *Suppose that S is a regular F-finite ring and that $R = S/I$. Suppose that $Q \in \operatorname{Spec} S$ contains I. Then Q/I is a non-F-regular center of R if and only if $I^{[p^e]} : I \subseteq Q^{[p^e]} : Q$ for all $e \geq 0$. Furthermore, if R/I is $\mathbb{Q}$-Gorenstein such that $(p^e - 1)K_R$ is Cartier, then one may check that single $e > 0$.*

On the other hand if S is sufficiently local and if a $\mathbb{Q}$-divisor Δ on $\operatorname{Spec} R$ corresponds to a map $\phi : F^e_* R \to R$. Fix $d \in S \in I^{[p^e]} : I$ corresponding to ϕ. Then Q/I is a non-F-regular center of (R, Δ) if and only if $d \in Q^{[p^e]} : Q$.

Proof. The statements are local, so we may assume that S is local. But then the result follows immediately one recalls that $F^e_*(I^{[p^e]} : I)$ maps surjectively onto $\operatorname{Hom}_R(F^e_* R, R)$. □

Remark 18.12. For a log canonical pair (X, Δ), the set of LC-centers satisfy many remarkable properties. For example, if the pair (X, Δ) is log canonical:

- Any union of such centers is seminormal (and in fact, Du Bois).
- Any intersection of such centers is a union of such centers.

F-pure centers satisfy the analogous results.

One can certainly ask if other natural properties of LC-centers hold for F-pure centers. The set of LC-centers are finite for a log canonical pair, so we can ask the following.

Theorem 18.13. [Sch09], [MK09] *If (X, Δ) is sharply F-pure, then there are finitely many F-pure centers.*

Proof. Choose ϕ such that $\Delta_\phi \geq \Delta$ and that $\phi(1) = 1$. Note, every center of sharp F-purity $Q \in \operatorname{Spec} R$ for (R, Δ) satisfies $\phi(F^e_* Q) \subseteq Q$. We will show that there are finitely many prime ideals Q such that $\phi(F^e_* Q) \subseteq Q$. First note that if there are infinitely many such prime ideals, one can find a collection $\mathfrak{Q}$ of infinitely many centers which all have the same height and whose closure (in the Zariski topology) is an irreducible subscheme of $\operatorname{Spec} R$ with generic point P (ie, P is the minimal associated prime of $\bigcap_{Q \in \mathfrak{Q}} Q$). Notice P must have smaller height than the elements of $\mathfrak{Q}$. Notice further that P also satisfies $\phi(F^e_* P) \subseteq P$ since it is the intersection of the elements of $\mathfrak{Q}$ (in other words, P is a center of sharp F-purity for (R, Δ_ϕ)).

By restricting to an open set, we may assume that R/P is normal (the elements of $\mathfrak{Q}$ will still form a dense subset of $V(P)$). Then ϕ induces a divisor Δ_P on $\operatorname{Spec} R/P$ as above. But the set of elements in $\mathfrak{Q}$ restrict to centers of sharp F-purity for $(R/P, \Delta_P)$ by F-adjunction. As noted above, $\{Q/P \mid Q \in \mathfrak{Q}\}$ is dense in $\operatorname{Spec} R/P$ and simultaneously $\{Q/P | Q \in \mathfrak{Q}\}$ is contained in the non-strongly F-regular locus of $(R/P, \Delta_P)$, which is closed and proper. This is a contradiction. $\square$

19. F-rationality via alterations and finite maps

In this section, we will show that F-rationality can also be described via alterations. First we prove the *equational lemma*, which lets us kill cohomology in characteristic $p > 0$ by passing to finite covers, the variant we give appeared recently in the work of Huneke-Lyubeznik, but the result has connections to the work of Hochster and others even in the 70s, as well as the work of Hochster-Huneke and Smith.

Theorem 19.1 (Equational-Lemma)**.** [HL07], [HH92] *Let R be a commutative Noetherian domain containing a field of characteristic $p > 0$. Let K be the fraction field of R and suppose that $\bar{K}$ is the algebraic closure of K. Let I be an ideal of R and suppose that $\alpha \in H^i_I(R)$ is an element such that $\alpha, \alpha^p, \alpha^{p^2}, \dots$ belong to a finitely generated R-submodule of $H^i_I(R)$. Then there exists an R-subalgebra R' of $\bar{K}$ that is a finite R-module and such that the induced map $H^i(R) \to H^i(R')$ sends α to zero.*

This proof is taken from [HL07]*.* Let A_t denote the submodule generated by $\alpha, \alpha^p, \dots, \alpha^{p^t}$. By hypothesis, $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ eventually stabilizes at A_s (note we may take s not divisible by p). Thus we have an equation:

$$g(T) = T^{p^s} - r_1 T^{p^{s-1}} - r_2 T^{p^{s-2}} - \cdots - r_{s-1} T$$

where $r_i \in R$ and for which α is a root. It is a key point here that g is additive in T (because of the pth powers).

Suppose that $x_1, \dots, x_n$ generate I and consider the Čech complex

$$0 \longrightarrow C^0(M) \xrightarrow{d_0} C^1(M) \xrightarrow{d_1} C^2(M) \longrightarrow \dots \xrightarrow{d_{n-1}} C^n(M) \longrightarrow 0$$

where $C^0(M) = M$ and $C^i(M) = \oplus_{j_1 \leq \dots \leq j_i} M_{x_{j_1} \dots x_{j_i}}$ (we will set $M = R$ and also equal to certain finite extensions of R).

Suppose that $\widetilde{\alpha} \in C^i(R)$ is a cycle that represents α. We know that $g(\widetilde{\alpha}) = d_{i-1}(\beta) \in d_{i-1}(C^{i-1}(R))$ since $g(\alpha) = 0$. Write

$$\beta = \oplus_{j_1 \leq \dots \leq j_{i-1}} \left(\frac{r_{j_1 \dots j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e} \right)$$

for some (uniform) integer e.

For each tuple $j_1 \leq \cdots \leq j_i$, consider the equation

$$g\left(\frac{Z_{j_1 \leq \dots \leq j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e}\right) - \frac{r_{j_1 \dots j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e} = 0$$

in the variable $Z_{j_1 \leq \dots \leq j_{i-1}}$. Clearing denominators gives us monic polynomials $h_{j_1 \leq \dots \leq j_{i-1}}$ in the variables $Z_{j_1 \leq \dots \leq j_{i-1}}$. Let $z_{j_1 \leq \dots \leq j_{i-1}} \in \bar{K}$ be a root of this equation. Set R'' to be the finite extension of R generated by all the $z_{j_1 \leq \dots \leq j_{i-1}}$.

Set

$$\widetilde{\widetilde{\alpha}} = \oplus \left(\frac{z_{j_1 \leq \cdots \leq j_{i-1}}}{(x_{j_1} \ldots x_{j_{i-1}})^e} \right) \in C^{i-1}(R'')$$

We also know that $C^\bullet(R)$ is a subcomplex of $C^\bullet(R'')$ and so we can identify $\widetilde{\alpha}$ and β with their natural images in $C^\bullet(R'')$. Thus $\widetilde{\alpha} \in C^i(R'')$ is a cycle representing the image of α under $H^i_I(R) \to H^i_I(R'')$. As is $\bar{\alpha} = \widetilde{\alpha} - d_{i-1}(\widetilde{\widetilde{\alpha}})$ (we just subtracted a boundary, which does not change the cohomology class). Now, $g(\widetilde{\widetilde{\alpha}}) = \beta$ and also $g(\widetilde{\alpha}) = d_{i-1}(\beta)$, so that

$$g(\bar{\alpha}) = g(\widetilde{\alpha} - d_{i-1}(\widetilde{\widetilde{\alpha}})) = g(\widetilde{\alpha}) - g(d_{i-1}(\widetilde{\widetilde{\alpha}})) = g(\widetilde{\alpha}) - d_{i-1}(g(\widetilde{\widetilde{\alpha}})) = d_{i-1}(\beta) - d_{i-1}(\beta) = 0.$$

Write

$$\bar{\alpha} = \oplus \rho_{j_1 \leq \cdots \leq j_i} \text{ with } \rho_{j_1 \leq \cdots \leq j_i} \in R''_{j_1 \ldots j_i}.$$

We know that $g(\rho_{j_1 \leq \cdots \leq j_i}) = 0$ individually so that $\rho_{j_1 \leq \cdots \leq j_i}$ is integral over R. Set R' to be R'' adjoin the $\rho_{j_1 \leq \cdots \leq j_i}$ (this is contained in the normalization of R'').

By hypothesis, the image of α in $H^i_I(R')$ is represented by $\bar{\alpha} = \oplus \rho_{j_1 \leq \cdots \leq j_i}$. We need to show that this is a boundary. However, there is an exact subcomplex of $C^\bullet(R')$ which is simply R' in each term, $\bar{\alpha}$ is certainly in this subcomplex and thus it is a boundary as desired. $\square$

Before continuing on, we need a very brief introduction to Matlis/local-duality. Suppose that $(R, \mathfrak{m})$ is a local ring. We know every R-module lives inside an injective R-module. In particular, $R/\mathfrak{m}$ lives inside an injective R-module I. It turns out that there is in some sense a smallest (up to containment) injective module E containing $R/\mathfrak{m}$. This module is unique up to isomorphism and is called the injective hull of $R/\mathfrak{m}$ and will be denoted by $E = E_{R/\mathfrak{m}}$.

Theorem 19.2 (Matlis)**.** [BH93] $\mathrm{Hom}_R(\_\_, E)$ *is an exact functor which (faithfully) takes finitely generated R-modules to artinian R-module. Furthermore, if R is complete, then the functor (faithfully) takes artinian R-modules to finitely generated R-modules, induces an equivalence of categories between the two sets, and applying it twice is an isomorphism. Finally,* $\mathrm{Hom}_R(\_\_, E)$ *always induces an equivalence of the category of finite length R-modules (i.e. , Noetherian + Artinian modules).*

Theorem 19.3 (Grothendieck)**.** [Har66] *With notation as above,* $\mathrm{Hom}_R(h^{-i}(\omega_R^\bullet), E) \cong h^{-i}(\mathbf{R}\,\mathrm{Hom}_R(\omega_R^\bullet, E)) = H^i_{\mathfrak{m}}(R)$. *More generally for $M \in D^b_{\mathrm{coh}}(R)$ there is a functorial isomorphism*

$$\mathrm{Hom}_R(h^{-j}\mathbf{R}\,\mathrm{Hom}_R(M, \omega_R^\bullet), E) \cong \mathbb{H}^j_{\mathfrak{m}}(M).$$

Corollary 19.4. *With notation as above,* $\mathrm{Hom}_R(F^e_* \omega_R, E) \cong H^d_{\mathfrak{m}}(F^e_* R)$.

Corollary 19.5. *An F-finite ring R is F-rational if and only if it is:*

(a) *R is Cohen-Macaulay, and*

(b) *for every finite extension* $R \subseteq S$*, the natural map* $T : \omega_S \to \omega_R$ *is surjective.*

Condition (b) and also be replaced by

(b*) *for every generically finite proper map* $\pi : Y \to \operatorname{Spec} R$*, the natural map* $T : \pi_* \omega_Y \to \omega_R$ *is surjective, or*

(b**) *for every alteration* $\pi : Y \to \operatorname{Spec} R$*, the natural map* $T : \pi_* \omega_Y \to \omega_R$ *is surjective.*

Proof. It is harmless to assume that R is normal (otherwise the normalization map breaks condition (b), (b*) and (b**)).

First we will show that F-rational implies (b*) (which obviously implies (b) and (b**)). But this is easy, simply consider the commutative diagram

$$\begin{array}{ccc} F_* \pi_* \omega_Y & \xrightarrow{\pi_* \Psi_Y} & \pi_* \omega_Y \\ \downarrow T & & \downarrow T \\ F_* \omega_R & \xrightarrow[\Psi_R]{} & \omega_R \end{array}$$

The image of T is clearly Ψ_R-stable and non-zero, and F-rational implies that there are no proper Ψ_R-stable submodules.

Conversely, suppose we have conditions (a) and (b) (note that condition (b) is automatically implied by (b*) and (b**)). Suppose that R is not F-rational. By localizing at the generic point of the non-F-rational locus, we may assume that $(R, \mathfrak{m})$ is a local d-dimensional ring which is F-rational on the punctured spectrum. This means that $\omega_R / \tau(\omega_R)$ is supported at the maximal ideal. We set E to be an injective hull of $R/\mathfrak{m}$ and apply $\operatorname{Hom}_R(\underline{\ \ }, E)$ to the short exact sequence:

$$0 \to \tau(\omega_R) \to \omega_R \to \omega_R / \tau(\omega_R) \to 0$$

yielding

$$0 \leftarrow \tau(\omega_R)^\vee \leftarrow H^d_\mathfrak{m}(R) \leftarrow (\omega_R / \tau(\omega_R))^\vee \leftarrow 0.$$

We knew that $\tau(\omega_R)$ is $\Phi_R : F_* R \to R$ stable. It follows that its dual is stable under the Frobenius action $H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(F^e_* R)$. Its dual is a finite length Frobenius stable submodule, thus there exists a finite extension $R \subseteq S$ such that $(\omega_R / \tau(\omega_R))^\vee$ is sent to zero in $H^d_\mathfrak{m}(S)$. Consider the diagram:

$$\omega_S \to \omega_R \to \omega_R / \operatorname{Image}(\omega_S) \to 0$$

The dual is

$$H^d_\mathfrak{m}(S) \leftarrow H^d_\mathfrak{m}(R) \leftarrow K \leftarrow 0.$$

We know that $(\omega_R / \tau(\omega_R))^\vee$ is contained in K. Thus $\tau(\omega_R) = \operatorname{Image}(\omega_S \to \omega_R)$. $\square$

Remark 19.6. The submodule $(\omega_R / \tau(\omega_R))^\vee \subseteq H^d_\mathfrak{m}(R)$ is often denoted by $0^*_{H^d(R)}$ and is called the *tight closure of zero in* $H^d_\mathfrak{m}(R)$.

The proof leads us to the following question. Does there always exist a finite map $R \subseteq S$ such that $\tau(R) = \mathrm{Image}(\omega_S \to \omega_R)$?

Theorem 19.7. [HL07]*[cf Hochster-Yao] Suppose R is an F-finite domain. Then there always exists a finite map $R \subseteq S$ such that $\tau(R) = \mathrm{Image}(\omega_S \to \omega_R)$ and therefore*

$$\tau(\omega_R) = \bigcap_{R \subseteq S} \mathrm{Image}(\omega_S \to \omega_R).$$

More generally,

$$\tau(\omega_R) = \bigcap_{f : Y \to \operatorname{Spec} R \text{ a regular alteration}} \mathrm{Image}(f_* \omega_Y \to \omega_R).$$

Proof. The statement is local so we assume that R is a local ring with maximal ideal $\mathfrak{m}$.

First we show that the second statement follows from the first. To do this, we simply observe that $\tau(\omega_R) \subseteq \mathrm{Image}(f_* \omega_Y \to \omega_R)$ for any generically finite proper dominant map $f : Y \to \operatorname{Spec} R$ (this is based on the usual argument used to prove "F-rational ($\Rightarrow$) rational", which is essentially due to K. Smith, [Smi97a]).

To see this, consider the diagram

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[d, "f"'] & Y \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "F"] & \operatorname{Spec} R \end{tikzcd}$$

where the horizontal arrows are Frobenius. It is an easy application of Grothendieck duality that we have a commutative diagram:

$$\begin{tikzcd} F_* f_* \omega_Y \arrow[r, "f_* \Phi_Y"] \arrow[d] & f_* \omega_Y \arrow[d] \\ F_* \omega_R \arrow[r, "\Phi_R"] & \omega_R \end{tikzcd}$$

The image of $f_* \omega_Y \to \omega_R$ is non-zero at every maximal dimensional component of $\operatorname{Spec} R$ since f is generically finite and dominant. From this diagram and the definition of the parameter test submodule it immediately follows that $\mathrm{Image}(f_* \omega_Y \to \omega_R)$ contains $\tau(\omega_R)$.

It hence remains to show that we can find some finite map where the containment is indeed equality. For this we closely follow the strategy of [HL07]: Choose $\eta \in \operatorname{Spec} R$ to be a generic point of the non-F-rational locus of R. We know that $(\omega_R/\tau(\omega_R))^\vee = 0^*_{\mathbb{H}^d_{\mathfrak{m}}(R)}$ (where $(\cdot)^\vee$ denotes the Matlis dual by []. Because the punctured spectrum of R is F-rational, $\omega_R/\tau(\omega_R)$ and thus also $0^*_{\mathbb{H}^d_{\mathfrak{m}}}(R)$ has finite length.

It follows from the equational lemma above, that there exists a finite extension of reduced rings $R_\eta \subseteq S_\eta$ such that the image of $0^*_{\mathbb{H}_\eta^{\dim R_\eta}(R_\eta)}$ in $H_\eta^{\dim R_\eta}(S_\eta)$ is zero. By taking the normalization of R inside the total field of fractions of S_η, we may assume that S_η is indeed the localization of some S' at η. Because $0^*_{\mathbb{H}_\eta^{\dim R_\eta}(R_\eta)} \to H_\eta^{\dim R_\eta}(S_\eta)$ is zero, the Matlis dual map $(\omega_S)_\eta \to (\omega_R/\tau(\omega_R))_\eta$ is also zero.

By doing this for each generic point of the non-F-rational locus of R and taking a common extension, we can set S^1 to be a common extension of all the S'. We thus have $I_1 := \mathrm{Image}(\omega_{S^1} \to \omega_R)$ such that the support of $I_1/\tau(\omega_R)$ is of strictly smaller dimension than the non-F-rational locus. Choose $\eta_1 \in \operatorname{Spec} S^1$, a generic point of that support and suppose that $\dim R_{\eta_1} = d_1$. Therefore, $(I_1/\tau(\omega_R))_{\eta_1}$ has finite length.

Consider the map $g_1 : H^{d_1}_{\eta_1}(R_{\eta_1}) \to H^{d_1}_{\eta_1}(S^1_{\eta_1})$ and note that the image of $\omega_{S^1})_{\eta_1} \to \omega_R/\tau(\omega_R)$ has support at η_1 which implies that the image of $g_1(0^*_{H^{d_1}_{\eta_1}(R_{eta_1})}) \subseteq H^{d_1}_{\eta_1}(S^1_{\eta_1})$ is also finite length.

Choose z in that image. We know $z, z^p, z^{p^2}, \ldots$ are also contained in $g_1(0^*_{H^{d_1}(R_{eta_1})})$ since $0^*_{H^{d_1}_{\eta_1}(R_{\eta_1})}$ is stable under the Frobenius action. Therefore, there exists a finite extension $S^1_{\eta_1} \subseteq S''^1_{\eta_1}$ such that z is sent to zero under the map $H^{d_1}_{\eta_1}(S^1_{\eta_1}) \to H^{d_1}_{\eta_1}(S''^1_{\eta_1})$. Because $(\tau(\omega_{R_{\eta_1}}))^\vee/\ker(g_1)$ is finite length, we can find a common extension $S_{\eta_1} \subseteq S'^1_{\eta_1}$ which kills $0^*_{H^{d_1}_{\eta_1}(R_{\eta_1})}$. Thus the map $\omega_{S'_{\eta_1}} \to \omega_{R_{\eta_1}}$ has image $\tau(\omega_{R_{\eta_1}})$. Set S^2 to be the normalization of R inside the fraction field of S'^1. Define $I_2 := \mathrm{Image}\,(\omega_{S^2} \to \omega_R)$.

It follows that $(I_2/\tau(\omega_R))$ has support a strictly smaller closed subset than $(I_1/\tau(\omega_R))$. Continuing in this way, Noetherian induction tells us that eventually $\tau(\omega_R)$ is the image of some map $\omega_{S^n} \to \omega_R$. $\square$

20. Vanishing theorems via finite maps and direct summand conditions

Using the methods discussed previously, one can show the following.

Proposition 20.1. [HH92] *[Also see* [Smi97c] *and erratum on Smith's web page] Suppose that X is a projective variety of characteristic $p > 0$ and that $\mathscr{L}$ is an ample line bundle. Then there exists a finite map $f : Y \to X$ such that $H^i(X, \mathscr{L}^{-j}) \to H^i(Y, f^*\mathscr{L}^{-j})$ is zero for all $0 < i < \dim X$ and all j.*

The only interesting part of this statement is the case when $j = 0$ (just take f to be a high power of the Frobenius), and the idea of the proof is the same as the equational lemma. Recently Bhargav Bhatt, see

http://www-personal.umich.edu/~bhattb/math/ddscposchar.pdf

has shown that we can extend this result in the following way

Theorem 20.2 (Bhatt)**.** *[http://www-personal.umich.edu/~bhattb/math/ddscposchar.pdf] Suppose that X is a projective variety of characteristic $p > 0$ and $\mathscr{L}$ is a semi-ample line bundle. Then there exists a finite map $f : Y \to X$ such that*

- *$H^i(X, \mathscr{L}) \to H^i(Y, f^*\mathscr{L})$ is zero for $i > 0$.*
- *If in addition, $\mathscr{L}$ is big, then we can force $H^i(X, \mathscr{L}^{-1}) \to H^i(Y, f^*\mathscr{L})$ to be zero for $i < \dim X$.*

I'll leave you to find the proofs on the web.

Bhatt was actually interested in the following. Consider the following condition on a ring R.

Definition 20.3. Suppose that R is F-finite normal domain. We say that R is a splinter (or DSCR = direct summand condition ring) if $R \subseteq S$ splits as a map of R-modules for every finite extension $R \subseteq S$. Furthermore, we say that R is a DDSCR (= derived direct summand condition ring) if $R \subseteq Rf_*\mathcal{O}_Y$ splits as a map of objects in $D^b_{\mathrm{coh}}(R)$ for every generically finite proper map $f : Y \to \operatorname{Spec} R$.

Bhatt's main result follows:

Theorem 20.4 (Bhatt)**.** *A ring in characteristic $p > 0$ is a DSCR (= splinter) if and only if it is a DDSCR.*

Again, I'll refer you to his paper for the reference.

The following is the most important conjecture in tight closure theory (or a variant of it).

Conjecture 20.5. *A ring R satisfies the DSCR if and only if it is strongly F-regular.*

This conjecture is known in the $\mathbb{Q}$-Gorenstein case, see [Sin99a], [HH94b].

The implication that strongly F-regular implies DSCR is easy, we prove it below.

Lemma 20.6. *Suppose that R is strongly F-regular, then R is a splinter/DSCR.*

Proof. Given a finite extension $R \subseteq S$, fix $\phi : F^e_* R \to R$. This map induces a map $\operatorname{Hom}_R(S, F^e_* R) \to \operatorname{Hom}_R(S, R)$. We also have

$$F^e_* \operatorname{Hom}_R(S, R) = \operatorname{Hom}_{F^e_* R}(F^e_* S, F^e_* R) \to \operatorname{Hom}_R(F^e_* S, F^e_* R) \to \operatorname{Hom}_R(S, F^e_* R)$$

giving us a map $F^e_* \operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(S, R)$. One can check that this induces a commutative diagram.

$$\begin{tikzcd} F^e_* \operatorname{Hom}_R(S, R) \arrow[r] \arrow[d] & \operatorname{Hom}_R(S, R) \arrow[d] \\ F^e_* R \arrow[r, "\phi"'] & R \end{tikzcd}$$

where the vertical maps are evaluation-at-1. In particular, the image of $\operatorname{Hom}_R(S, R) \to R$ is ϕ-stable, and so if R is strongly F-regular, that map is surjective. $\square$

In mixed characteristic, one can ask a related question.

Conjecture 20.7 (Hochster)**.** *Suppose that R is a regular (local) ring in mixed characteristic. Is it true that for every finite extension $R \subseteq S$, one has that the evaluation-at-1 map $\mathrm{Hom}_R(S, R) \to R$ surjects (in other words, $R \subseteq S$ splits).*

This is probably the most important conjecture in commutative algebra. This conjecture is known up through dimension 3 and is closely related to a pantheon of other conjectures known as the homological conjectures. It is obvious in dimension 1 (1-dimensional regular local rings being PIDs). Let me prove it in dimension 2.

Proposition 20.8. *Suppose that R is a regular local ring and that $R \subseteq S$ is a finite extension. Then $R \subseteq S$ splits as a map of R-modules.*

Proof. It is sufficient to prove the result in the case that S is normal and reduced and so we assume that. Choose $f \in R$ such that R/f is regular (and 1-dimensional). Consider the following diagram (we do Elkik's proof yet again).

$$\begin{tikzcd}
\omega_S = \mathrm{Hom}_R(S,R) \arrow[r] \arrow[d, "\times f"] & R = \omega_R \arrow[d, "\times f"'] \\
\omega_S = \mathrm{Hom}_R(S,R) \arrow[r] \arrow[d] & R = \omega_R \arrow[d] \\
\omega_{S/f} \arrow[r] \arrow[d] & R/f = \omega_{R/f} \arrow[d] \\
0 & 0
\end{tikzcd}$$

where the bottom zeros exist because the rings in question are Cohen-Macaulay.

Now, $R/f \to S/f$ is a finite extension (S/f may not be reduced, but this doesn't matter), and so it splits because R/f is regular. Thus $\omega_{S/f} \to \omega_{R/f}$ surjects. In particular, our diagram becomes.

$$\begin{tikzcd}
\omega_S = \mathrm{Hom}_R(S,R) \arrow[r] \arrow[d, "\times f"] & R = \omega_R \arrow[r] \arrow[d, "\times f"'] & C \arrow[r] \arrow[d, "\times f"] & 0 \\
\omega_S = \mathrm{Hom}_R(S,R) \arrow[r] \arrow[d] & R = \omega_R \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
\omega_{S/f} \arrow[r] \arrow[d] & R/f = \omega_{R/f} \arrow[r] \arrow[d] & 0 & \\
0 & 0 & &
\end{tikzcd}$$

Nakayama's lemma again implies that C is zero. $\square$

Remark 20.9. In fact, one can show that for a regular ring in mixed characteristic for any generically finite map $f_*Y \to \operatorname{Spec} R$, $f_*\omega_Y \to \omega_R$ surjects.

21. Tight closure

Suppose that $R \subseteq S$ is an extension of rings. Consider an ideal $I \subseteq R$ and its extension IS. We always have that $(IS) \cap R \supseteq I$, however:

Lemma 21.1. *With $R \subseteq S$ as above and further suppose the extension splits as a map of R-modules. Then*

$$(IS) \cap R = I.$$

Proof. Fix $\phi : S \to R$ to be the splitting given by hypothesis. Suppose that $z \in (IS) \cap R$, in other words, $z \in IS$ and $z \in R$. Write $I = (x_1, \dots, x_n)$, we know that there exists $s_i \in S$ such that $z = \sum s_i x_i$. Now, $z = \phi(z) = \phi\left(\sum s_i x_i\right) = \sum x_i \phi(s_i) \in I$ as desired. □

A converse result holds too.

Theorem 21.2. [Hoc77] *Suppose that $R \subseteq S$ is a finite extension of approximately Gorenstein*[7] *rings. If for every ideal $I \subseteq R$, we have $IS \cap R = S$, then $R \subseteq S$ splits as a map of R-modules.*

Proof. See, [Hoc77] □

Consider now what happens if the extension $R \subseteq S$ is the Frobenius map.

Definition 21.3. Given an ideal $I \subseteq R$, the *Frobenius closure of I* (denoted I^F) is the set of all elements $z \in R$ such that $z^{p^e} \in I^{[p^e]}$ for some $e > 0$. Equivalently, it is equal to the set of all elements $z \in R$ such that $z \in (IR^{1/p^e})$ for some ideal I.

Remark 21.4. The set I^F is an ideal. Explicitly, if $z_1, z_2 \in I^F$, then $z_1^{p^a} \in I^{[p^a]}$ and $z_2^{p^b} \in I^{[p^b]}$. Notice that we may assume that $a = b$. Thus $z_1 + z_2 \in I^F$. On the other hand, clearly $hz_1 \in I^F$ for any $h \in R$.

We'll point out a couple basic facts about I^F.

Proposition 21.5. *Fix R to be a domain and $(x_1, \dots, x_n) = I \subseteq R$ an ideal.*

(i) $(I^F)^F = I^F$.
(ii) *For any multiplicative set W, $(W^{-1}I)^F = W^{-1}(I^F)$.*
(iii) *R is F-pure/split if and only if $I = I^F$ for all ideals $I \subseteq R$.*

Proof. For (i), suppose that $z \in (I^F)^F$. Thus there exists an $e > 0$ such that $z^{p^e} \in (I^F)^{[p^e]}$. In particular, we can write $z^{p^e} = \sum a_i x_i^{p^e}$ for some $a_i \in R$ and $x_i \in I^F$. Thus for each x_i, there exists an $e_i > 0$ such that $x_i^{p^{e_i}} \in I^{[p^{e_i}]}$.

[7]Nearly all rings in geometry satisfy this condition. Explicitly, a local ring $(R, \mathfrak{m})$ is called approximately Gorenstein if for every integer $N > 0$, there exists $I \subseteq \mathfrak{m}^N$ such that R/I is Gorenstein.

Choosing $e' \geq e_i$ for all i, we have that $x_i^{p^{e'}} \in I^{[p^{e'}]}$. Therefore, $(z^{p^e})^{p^{e'}} = z^{p^{e+e'}} = \sum a_i^{p^{e'}} x_i^{p^{e+e'}} \in I^{[p^{e+e'}]}$ as desired.

For (ii), we note that ($\supseteq$) is obvious. Conversely, suppose that $z \in (W^{-1}I)^F$, thus $z^{p^e} \in (W^{-1}I)^{[p^e]} = W^{-1}(I^{[p^e]})$. Therefore, for some $w \in W$, $wz^{p^e} \in I^{[p^e]}$, which implies that $(wz)^{p^e} \in I^{[p^e]}$ and the converse inclusion holds.

Part (iii) is obvious by Theorem 21.2. $\square$

It is natural to hope that these ideas can be extended to (strong) F-regularity.

Recall that R is strongly F-regular (a domain) if for each $0 \neq c \in R$, there exists a map $\phi : F^e_* R \to R$ that sends $c \mapsto 1$ for some $e > 0$.

Definition 21.6. [HH90] Suppose that R is an F-finite domain and I is an ideal of R, then the *tight closure of* I (denoted I^*) is defined to be the set

$$\{z \in R | \exists 0 \neq c \in R \text{ such that } cz^{p^e} \in I^{[p^e]} \text{ for all } e \geq 0\}.$$

Proposition 21.7. *Suppose we have an ideal* $(x_1, \dots, x_n) = I \subseteq R$ *where* R *is an* F*-finite domain.*

(i) I^* *is an ideal containing* I.
(ii) $(I^*)^* = I^*$.
(iii) *It is known that the formation of* I^* *does* NOT *commute with localization.*
(iv) *If* R *is strongly* F*-regular, then* $I^* = I$ *for all ideals* I.

Proof. For (i), suppose that $cz^{p^e} \in I^{[p^e]}$ and $dy^{p^e} \in I^{[p^e]}$ for all $e \geq 0$ for certain $c, d \in R \setminus \{0\}$. Then $cd(z+y)^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. Of course, clearly I^* contains I (choose $c = 1$).

For (iv), suppose that $z \in I^*$ and R is strongly F-regular. Choose $c \neq 0$ such that $cz^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. We know that there exists an $e > 0$ and $\phi : F^e_* R \to R$ which sends c to 1. Write $cz^{p^e} = \sum a_i x_i^{p^e}$. Then $z = \phi(cz^{p^e}) = \sum x_i \phi(a_i) \in I$. $\square$

Conjecture 21.8 (Weak$\Rightarrow$Strong)**.** *If* $I^* = I$ *for all ideals* $I \subseteq R$*, then* R *is strongly* F*-regular.*

Remark 21.9. This conjecture is known for $\mathbb{Q}$-Gorenstein rings (or even local rings which are $\mathbb{Q}$-Gorenstein on the punctured spectrum), for graded rings, and also for rings of finite type over an uncountable field.

Definition 21.10. A *finitistic test element* $0 \neq c \in R$, is an element of R such that for every ideal I and every $z \in I^*$,

$$cz^{p^e} \in I^{[p^e]}$$

for all $e \geq 0$.

It should be highly unclear that such a test element exists. However, we have already shown the following lemma.

Lemma 21.11. *12 Given an F-finite domain R, there exists $0 \neq c \in R$ such that for every $0 \neq d \in R$, $c \in \phi(dR)$ for some $\phi : F^e_* R \to R$.*

Corollary 21.12. *The c in the above lemma is a finitistic test element.*

Proof. Suppose that $0 \neq d \in R$ is an element of R such that $dz^{p^e} \in I^{[p^e]}$ for all $e > 0$, it follows from the statement above that there exists $\phi : F^a_* R \to R$ such that $\phi(d) = c$. Thus, for $e \geq a$,

$$cz^{p^e} = \phi(dz^{p^{e+a}}) \in \phi\left(I^{[p^{e+a}]}\right) \subseteq I^{[p^e]}.$$

□

Definition 21.13. The *finitistic test ideal* $\tau_f(R)$ is defined to be the ideal of R generated by all finitistic test elements. It can also be described as the set made up of all finitistic test elements and zero.

Lemma 21.14. *We have $\tau_f(R) = \cap_{I \subseteq R}(I : I^*)$.*

Proof. Suppose that $c \in \tau_f R$, then $cz^{p^e} \in I^{[p^e]}$ for all $e \geq 0$, in particular for $e = 0$. Thus $cz \in I$ and $c \in \cap_{I \subseteq R}(I : I^*)$.

Conversely, suppose that $c \in \cap_{I \subseteq R}(I : I^*)$. Choose $z \in I^*$. Then I claim that $z^{p^a} \in (I^{[p^a]})^*$ for all $a \geq 0$. But $cz^{p^e} \in I^{[p^e]}$ for all $e \geq 0$ so that $c^{p^a}(z^{p^a})^{p^e} \in (I^{[p^a]})^{[p^e]}$ for all a, and the claim is proven. Thus $cz^{p^a} \in I^{[p^a]}$ for all $a \geq 0$ because c was chosen in the intersection, which implies that c is a finitistic test element. □

Corollary 21.15. *R is weakly F-regular if and only if $\tau_f(R) = R$.*

We now come to the proof of Briançon-Skoda theorem via tight closure.

Theorem 21.16. [HH90] *Let R be an F-finite domain, and $(u_1, \dots, u_n) = I \subseteq R$ an ideal. Then for every natural number m,*

$$\overline{I^{m+n}} \subseteq \overline{I^{m+n-1}} \subseteq (I^m)^*$$

and so

$$\tau(R)\overline{I^{m+n}} \subseteq I^m.$$

which gives a very nice statement in the case that R is F-regular (and so $\tau(R) = R$).

This proof is taken from [Hoc07]. For any $y \in \overline{I^{m+n-1}}$, we know that there exists $0 \neq c \in R$ such that $cy^l \in (I^{m+n-1})^l$ for all $l \geq 0$. Consider a monomial $u_1^{a_1} \dots u_n^{a_n}$ where $a_1 + \dots + a_n = l(m+n-1)l$. Write each $a_i = b_i l + r_i$ where $0 \leq r_i \leq l-1$. We claim that the sum of the b_i is at least m, which will imply that the monomial is contained in $(I^m)^{[l]}$ for all l such that $l = p^e$. However, if the sum $b_1 + \dots + b_m \leq m-1$, then $l(m+n-1) = \sum a_i \leq l(m-1) + n(l-1) = l(m+n-1) - n < l(m+n-1)$, which implies the claim.

Thus $cy^{p^e} \in (I^m)^*$ as desired. □

Remark 21.17. Previously, in 17.8, we used this theorem on $\overline{\mathfrak{a}^{\lceil t(p^e-1)\rceil+r}}$ where r is the number of generators of $\mathfrak{a}$. The tight-closure Briancon-Skoda theorem tells us that this is contained in $\mathfrak{a}^{\lceil t(p^e-1)\rceil}$.

21.1. **Hilbert-Kunz(-Monsky) multiplicity.** Recall the following definition:

Definition 21.18. Suppose that $(R, \mathfrak{m})$ is a d-dimensional local ring and I is an $\mathfrak{m}$-primary ideal. We define the *multiplicity of R (at I)* to be

$$e(I, R) := \lim_{n\to\infty} \frac{d!(R/I^n)}{n^d}.$$

Note that R is regular if and only if $e(\mathfrak{m}, R) = 1$.

Using this as a guide, Kunz considered the following notion.

Definition 21.19. [Kun69a], [Mon83] Suppose that $(R, \mathfrak{m})$ is a d-dimensional local ring. We define the *Hilbert-Kunz-Monsky multiplicity of R (at $\mathfrak{m}$)* to be

$$e_{HKM}(I, R) := \lim_{n\to infty} \frac{(R/I^{[p^e]})}{p^{ed}}$$

Kunz showed that $e_{HKM}(\mathfrak{m}, R) = 1$ if R is regular (we basically also did in the first few days of class), and Watanabe-Yoshida [WY00] (and Huneke-Yao, [HY02]) showed the converse.

Remark 21.20. In fact, this $e(I, R)$ can be viewed as some sort of leading coefficient of a polynomial computing (R/I^n). While it is true that $(R/I^{[p^e]}) = e_{HKM}(I, R)p^{ed} + O(p^{e(d-1)})$, the lower order terms are not generally a polynomial, unlike $e(I, R)$

Kunz actually thought that this limit didn't exist, and even had a claimed counter-example. (Un?)Fortunately, there was a mistake and Monsky later showed that the limit did indeed exist. The reason we mention it now is the following theorem of Hochster-Huneke.

Theorem 21.21. [HH90] *Suppose $(R, \mathfrak{m})$ is an equidimensional F-finite local domain. Further suppose that $I \subseteq J$ are two $\mathfrak{m}$-primary ideals. Then if $J \subseteq I^*$ if and only if $e_{HKM}(I, R) = e_{HKM}(J, R)$.*

Proof. We will only prove one direction, for the converse, see [HH90]. Suppose then that $J \subseteq I^*$, in other words, suppose that $I^* = J^*$. We first show that there exists a $c \in R^\circ$ such that $cJ^{[q]} \subseteq I^{[q]}$ for all $q \gg 0$. But this is easy, choose a set of generators $x_1, \dots, x_k$ of J. Then by hypothesis, there exists a $c_i \in R$ such that $c_i x_i^q \in I^{[q]}$ for all $q \gg 0$. Let c be the product of the c_i and note that $cx_i^q \in I^{[q]}$ for all $q \gg 0$. Therefore, $J^{[q]}/I^{[q]}$ is a module with at most k generators over $R/(I^{[q]} + (c))$. Set $S = R/(c)$. Thus $J^{[q]}/I^{[q]}$ is a module with at most k generators over $S/(IS)^{[q]}$. Note that $\dim S \leq \dim R - 1$.

But now we know that there is a constant C_S such that $\lambda(S/(IS)^{[q]} \leq C_S q^{d-1}$ (since Hilbert-Kunz multiplicities exist). However, we can also map $(S/(IS)^{[q]})^{\oplus k}$ onto $J^{[q]}/I^{[q]}$. Therefore,

$$\lambda(J^{[q]}/I^{[q]}) \leq kC_S q^{d-1} h^{d-1}.$$

Thus $\lambda(R/J^{[q]}) - \lambda(R/I^{[q]}) \leq Cq^{d-1}$ for $C = kC_S h^{d-1}$.

Therefore the J and I have the same Hilbert-Kunz multiplicity. $\square$

22. Finitistic test ideals, tight closure for modules, and tight closure of pairs

Definition 22.1. [HH90] Given a domain R and R-modules $N \subseteq M$, we consider the natural map

$$\gamma_e : M \to M \otimes F^e_* R$$

for each e. We say that $z \in M$ is in the *tight closure of N in M* if there exists a $c \in R \setminus \{0\}$ such that for all $e \geq 0$, $\gamma_e(z).c = z \otimes c$ is contained in the image of $N \otimes F^e_* R \to M \otimes F^e_* R$.

Remark 22.2. Suppose that $M = R$ and N is an ideal. Then the image of $N \otimes_R F^e_* R$ inside $R \otimes_R F^e_* R = F^e_* R$ is simply $N^{[p^e]}$. Thus this definition of tight closure coincides with the usual one.

The case we are going to be primarily concerned with is when $N = 0 \subseteq M$. Generally speaking, one can always reduce to studying this case by the following trick.

Lemma 22.3. *Suppose $N \subseteq M$ is as above, then $z \in N^*_M$ if and only if $\bar{z} \in 0^*_{M/N}$.*

Proof. Now, $z \in N^*_M$ if and only if there exists $0 \neq c \in R$ such that

$$\gamma_e(z) \otimes c \in \text{Image}\left(N \otimes F^e_* R \to M \otimes F^e_* R\right).$$

But this happens if and only if $\gamma_e \bar{(z)} = 0 \subseteq (M/N) \otimes F^e_* R$ by right exactness of tensor. $\square$

Remark 22.4. In general, given $N \subseteq M \subseteq M'$, one has $N^*_M \subsetneq N^*_{M'}$. The problem is that $\otimes$ is not left-exact.

Lemma 22.5. *Suppose that R is strongly F-regular, then for every R-modules $N \subseteq M$, $N = N^*_M \subseteq M$.*

Proof. Suppose that $z \in N^*_M$. Thus there exists a $0 \neq d \in R$ such that $z \otimes d$ is contained in the image of $N \otimes F^e_* R \to M \otimes F^e_* R$ for all $e \geq 0$. Choose $\phi : F^a_* R \to R$ which sends $d \mapsto 1$. We have the following diagram

$$\begin{tikzcd} N \otimes F^a_* R \arrow[r, "\mathrm{id}_N \otimes \phi"] \arrow[d, "f"'] & N \arrow[d, hook, "g"] \\ M \otimes F^a_* R \arrow[r, "\mathrm{id}_M \otimes \phi"'] & M \end{tikzcd}$$

We know that $z \otimes d$ is in the image of f, let ζ be an element of $N \otimes F^a_* R$ which maps to it. Thus

$$g\left((\mathrm{id}_N \otimes \phi)(\zeta)\right) = (\mathrm{id}_M \otimes \phi)(z \otimes d) = z$$

But g is simply the inclusion of N into M which implies that $z \in N$ as desired. ☐

We also have the converse statement.

Proposition 22.6. [HH90], [Hoc07] *Suppose R is an F-finite local domain and that for every R-module $N \subseteq M$, $N = N^*_M$, then R is strongly F-regular.*

Proof. Let E denote the injective hull of the residue field $R/\mathfrak{m}$. We know $0^*_E = 0$ by assumption. We will show that R is strongly F-regular.

By hypothesis, $0^*_E = 0$. Choose $c \in R = F^e_* R$ and consider the map $R \to F^e_* R$ which sends $1 \mapsto c$. Tensoring with E, gives us a map $\gamma_{e,c} : E \to E \otimes_R F^e_* R$ which sends z to $z \otimes c$. Now recall that we have an isomorphism $F^e_* R \otimes \mathrm{Hom}(R, E) \cong F^e_* R \otimes_R E \cong \mathrm{Hom}_R(\mathrm{Hom}_R(F^e_* R, R), E)$ defined by the map which sends $r \otimes \phi$ to the map $h : \mathrm{Hom}_R(F^e_* R, R) \to E$ defined by the rule $h(\alpha) = \phi(\alpha(r))$. Thus $E \to E \otimes_R F^e_* R$ is identified with

$$E \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R, R), E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(F^e_* R, R), E).$$

The map is just induced by the inclusion $R \subseteq F^e_* R$ in the first entry which sends 1 to c. Apply $\mathrm{Hom}_R(\_\_, E)$ and Matlis duality. This gives us a map $\mathrm{Hom}_{\hat{R}}(F^e_* \hat{R}, \hat{R}) \to \mathrm{Hom}_{\hat{R}}(\hat{R}, \hat{R}) \cong \hat{R}$ induced by evaluation at c. In particular, $\gamma_{e,c}$ is injective if and only if the evaluation-at-c-map $\mathrm{Hom}_R(F^e_* R, R) \to R$ is surjective (we can remove the completion signs due to faithful flatness).

Consider now $c = 1$, we know that for any $z \in E$, $0 \neq z \otimes 1 \in E \otimes F^e_* R$ for infinitely many $e > 0$. But if it holds for infinitely many $e > 0$, then it holds for all $e \geq 0$ since $\gamma_{e,1}$ factors through $\gamma_{e-1,1}$. Therefore, $\gamma_{e,1}$ is injective and R is F-split.

Now, again consider $\gamma_{e,c}$. $\gamma_{e,c}$ is injective if and only if it is non-zero on the socle[8] Suppose that $z \in \ker(\gamma_{e,c})$, in other words $0 = z \otimes c \in E \otimes F^e_* R$. We claim that then also $z \in \ker(\gamma_{e-1,c})$. However, the composition

$$\begin{array}{ccccc} E & \xrightarrow{g} & E \otimes F^{e-1}_* R & \xrightarrow{f} & E \otimes F^e_* R \\ z & \longmapsto & z \otimes c & \longmapsto & z \otimes c^p, \end{array}$$

is certainly zero, and since the map f is injective (because R is F-split), this implies that $g(z) = 0$.

Therefore, the set of kernels of $\gamma_{e,c}$ are a descending sequence of modules in E, an artinian module. Therefore they eventually stabilize. However, no

[8]The 1-dimensional submodule of E which is annihilated by $\mathfrak{m}$.

element is in all the kernels because $0^*_E = 0$. Thus some evaluation-at-c-map $\mathrm{Hom}_R(F^e_* R, R) \to R$ is surjective, proving that R is strongly F-regular. $\square$

Generally speaking, using the same method as above, one can show that $\mathrm{Ann}_R\, 0^*_E = \tau(R)$, see for example [LS01]. In fact, any non-zero element of $\tau(R)$ can be used to "test" tight closure in any module. Furthermore, $\tau(R)$ is generated by exactly the elements $c \in R$ such that $cN^*_M \subseteq N$ for all modules $N \subseteq M$, see [Hoc07].

Conjecture 22.7. *The (big/non-finitistic) test ideal $\tau(R)$ is equal to the finitistic test ideal $\tau_f(R)$.*

Let us prove another variant of this below, first however, a lemma.

Lemma 22.8. *Suppose that R is a d-dimensional F-finite local domain. Then $H^d_{\mathfrak{m}}(R) \otimes F^e_* R$ is naturally identified with $H^d_{\mathfrak{m}}(F^e_* R)$.*

Proof. Choose a system of parameters $x_1, \dots, x_d$ for R, and compute local cohomology in terms of the Čech complex with respect to those parameters. $H^d_{\mathfrak{m}}(R)$ is then identified with the cokernel of the map

$$\oplus R_{\hat{x_i}} \to R_{x_1 \dots x_d}.$$

Tensoring that map with $F^e_* R$, gives us the term of the Čech complex corresponding to the system of parameters $x_1^{p^e}, \dots, x_d^{p^e}$. This completes the proof, in fact one also sees that $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R) \otimes F^e_* R$ is identified with $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_* R)$. $\square$

Proposition 22.9. [Smi97a] *Suppose that R is a d-dimensional F-finite local domain. Then the tight closure of zero in $H^d_{\mathfrak{m}}(R)$ is the unique largest non-zero module $M \subseteq H^d_{\mathfrak{m}}(R)$ such that $F(M) \subseteq M$ where $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R) = F_* H^d_{\mathfrak{m}}(R) = H^d_{\mathfrak{m}}(F_* R)$ is the map induced by Frobenius.*

Proof. For simplicity, we assume that R is complete, in the general case use the faithful flatness of $\mathrm{Hom}_R(\_\_, E)$. First we show that $F(0^*_{H^d_{\mathfrak{m}}(R)}) \subseteq 0^*_{H^d_{\mathfrak{m}}(R)}$. Suppose that $z \in 0^*_{H^d_{\mathfrak{m}}(R)}$. Thus there exists $c \in R$ such that $0 = cz^{p^e} \in H^d_{\mathfrak{m}}(R) \otimes F^e_* R$ for all $e \geq 0$ (by the previous lemma, we need not be careful about tensor products). Then $0 = c^p (z^p)^{p^e} \in H^d_{\mathfrak{m}}(R)$, so $F(z) \in 0^*_{H^d_{\mathfrak{m}}(R)}$.

Now suppose that N is any proper submodule of $H^d_{\mathfrak{m}}(R)$ such that $F(N) \subseteq N$. We know that $T := \mathrm{Hom}_R(H^d_{\mathfrak{m}}(R)/N, E) \subseteq \mathrm{Hom}_R(H^d_{\mathfrak{m}}(R), E) = \omega_R$. But ω_R is rank-one, so there exists a $c \in R$ such that $c\omega_R \subseteq T$, thus we have the composition

$$c\omega_R \subseteq T \subseteq \omega_R.$$

Dualizing again, we get

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)/N \to cH^d_{\mathfrak{m}}(R)$$

where the composition is multiplication by c. This implies that N is annihilated by c. Thus if $z \in N$, $cz^{p^e} = cF^e(z) \in cF^e(N) \subseteq cN = 0$ for all $e \geq 0$, implying that $z \in 0^*_{H^d_{\mathfrak{m}}(R)}$ and completing the proof. $\square$

Finally, we briefly define tight closure of pairs.

Definition 22.10. [Tak04b], [HY03], [Sch08b], [Sch08a], [HH90] Suppose R is an F-finite domain, $X = \operatorname{Spec} R$ and $(X, \Delta, \mathfrak{a}^t)$ is a triple. Further suppose that M is a (possibly non-finitely generated) R-module and that N is a submodule of M. We say that an element $z \in M$ is in the $(\Delta, \mathfrak{a}^t)$-*tight closure of* N *in* M, denoted $N_M^{*\Delta, \mathfrak{a}^t}$, if there exists an element $0 \neq c \in R$ such that, for all $e \gg 0$ and all $a \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$, the image of z via the map

$$(F^e_* i) \circ \mathbb{F}^e_*(\times ca) \circ F^e : M \longrightarrow M \otimes_R F^e_* R \xrightarrow{F^e_*(\times ca)} M \otimes_R F^e_* R \longrightarrow M \otimes_R F^e_* R(\lceil (p^e-1)\Delta \rceil)$$

is contained in $N_M^{[q]\Delta}$, where we define $N_M^{[q]\Delta}$ to be the image of $N \otimes_R F^e_* R(\lceil (p^e-1)\Delta\rceil)$ inside $M \otimes_R F^e_* R(\lceil (p^e-1)\Delta \rceil)$.

Most of the theory of test elements / ideals can be generalized to this setting, although some of the arguments used so far do not work. See [HY03], [Tak04b], [Sch08b] and [Sch08a] for some additional discussion.

23. Hara's surjectivity lemma

Our goal is to show the following theorem.

Lemma 23.1. [Har98] *Suppose that R_0 is a ring of characteristic zero, $\pi : \widetilde{X}_0 \to \operatorname{Spec} R_0$ is a log resolution of singularities, D_0 is a π-ample $\mathbb{Q}$-divisor with simple normal crossings support. We reduce this setup to characteristic $p \gg 0$. Then the natural map*

$$(F^e)^\vee = \Phi_{\widetilde{X}} : \pi_* F^e_* \omega_{\widetilde{X}}(\lceil p^e D \rceil) \to \pi_* \omega_{\widetilde{X}_p}(\lceil D \rceil)$$

surjects.

We will show it in the following way. We follow Hara's proof.

Proposition 23.2. *Suppose that X is a d-dimensional smooth variety (quasi-projective) of finite type over a perfect field k of characteristic $p > 0$.* [9] *Further suppose that $E = \sum E_j$ is a reduced simple normal crossings divisor on X. Suppose in addition that D is a $\mathbb{Q}$-divisor on X such that* $\operatorname{Supp}(D - \lfloor D \rfloor) = \operatorname{Supp}(\{D\}) \subseteq \operatorname{Supp}(E)$.

Additionally, suppose that the following two vanishings hold:

(a) $H^j(X, \Omega^i_X(logE)(-E - \lfloor -D \rfloor)) = 0$ *for* $i + j = d + 1$ *and* $j > 1$.

(b) $H^j(X, \Omega^i_X(logE)(-E - \lfloor -pD \rfloor)) = 0$ *for* $i + j = d$ *and* $j > 0$.

Then, the natural map

$$\begin{array}{rl} & H^0(X, F_* \omega_X(\lceil pD \rceil)) \\ = & \operatorname{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X(\lfloor -pD \rfloor), \omega_X) \\ \to & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(\lfloor -D \rfloor), \omega_X) \\ = & H^0(X, \omega_X(\lceil D \rceil)) \end{array}$$

[9] We may as well assume $k = \mathbb{F}_p$ for simplicity, we'll only want this for finite fields, and all the arguments are essentially the same as over $\mathbb{F}_p$.

surjects.

Our plan is as follows:

(i) Prove the proposition.
(ii) Show for an ample $\mathbb{Q}$-divisor D reduced from characteristic $p \gg 0$, conditions (a) and (b) hold.
(iii) The e-iterated version of Hara's lemma will then follow from composing the surjectivity from the proposition and composition of maps.

In order to prove the proposition, we will need to briefly recall the Cartier operator. From here on out, X and E are as in Proposition 23.2. Consider the (log)de-Rham complex, $\Omega^{\bullet}_X(\log E)$. This is not a complex of $\mathcal{O}_X$-modules (the differentials are not $\mathcal{O}_X$-linear). However, the complex

$$F_*\Omega^{\bullet}_X(\log E)$$

is a complex of $\mathcal{O}_X$-modules (notice that $d(x^p) = 0$).

Definition-Proposition 23.3. [Car57], [Kat70] [cf [EV92], [BK05]] There is a natural isomorphism (of $\mathcal{O}_X$-modules):

$$C^{-1} : \Omega^i_X(\log E) \to \mathcal{H}^i(F_*\Omega^{\bullet}_X(\log E))$$

Furthermore, $(C^{-1})^{-1}$ for $i = d$ and $E = 0$, induces a map $F_*\omega_X \to \mathcal{H}^d(F_*\Omega^{\bullet}_X(\log E)) \cong \omega_X$ which corresponds to the natural dual of Frobenius[10].

Let us explain how to construct this isomorphism C^{-1}. We follow [EV92, 9.13] and [Kat70]. We begin with C^{-1} in the case that $i = 1$ and $E = 0$. We work locally on X (which we assume is affine) and we define C^{-1} by its action on $dx \in \Omega^i_X(\log E)$, $x \in \mathcal{O}_X$; $C^{-1}(dx) = x^{p-1}dx$ (or rather, its image in cohomology). In the $E \neq 0$ case, if t is a local parameter of E, then we define $C^{-1}(\frac{dt}{t}) = dt/t$.

We should show that C^{-1} is additive, we start in the $E = 0$ case. First notice that $d(x^{p-1}dx) = 0$ so at least the image of $x^{p-1}dx$ is in the cohomology of the de Rham complex.

Now, $C^{-1}(d(x) + d(y)) = C^{-1}(d(x + y)) = (x + y)^{p-1}d(x + y)$, we need to compare this to $x^{p-1}dx + y^{p-1}dy$. Write $f = \frac{1}{p}\left((x + y)^p - x^p - y^p\right)$ (where the $\frac{1}{p}$ just formally cancels out the ps in the binomial coefficients). Then

$$df = d\sum_{i,j>0,i+j=p} \gamma_i x^i y^{p-i} = \left(\sum_{i>0,j>0,i+j=p-1} \gamma_i i x^{i-1} y^{p-i}\right) dx + \left(\sum_{i>0,j>0,i+j=p-1} \gamma_i p - i x^i y^{p-i-1}\right) dy$$

where $\gamma_i = \frac{1}{p}\binom{p}{i} = \frac{(p-1)(p-2)\dots 1}{i!(p-i)!} = \frac{1}{p-i}\binom{p-1}{i} = \frac{1}{i}\binom{p-1}{p-i}$. Thus

$$df = (x + y)^{p-1}(dx + dy) - x^{p-1}dx - y^{p-1}dy.$$

Therefore, $x^{p-1}dx + y^{p-1}dy$ and $(x+y)^{p-1}d(x+y)$ are the same in cohomology.

[10]This is important, it gives us a "canonical" map between these two modules (before it was always defined up to multiplication by units)

For the $E \neq 0$ case and t a defining equation of a component of E, simply observe that

$$C^{-1}(dt) = C^{-1}\left(t\frac{dt}{t}\right) = t^p C^{-1}\left(\frac{dt}{t}\right) = t^p \frac{dt}{t} = t^{p-1}dt,$$

which at least shows that the definition of C^{-1} we gave is compatible, the additivity follows.

We define C^{-1} for $i > 1$ using wedge powers of C^{-1} for $i = 1$. We should also show that all these C^{-1} are isomorphisms. For simplicity, we work with the case that $X = \mathbb{F}_p[x, y]$ and $E = 0$ (see [EV92] or [Kat70] for how to reduce the polynomial ring case in general), let us explicitly see that the first C^{-1} is an isomorphism.

First we show that C^{-1} is injective. Suppose that $C^{-1}(fdx+gdy) = 0$, which means $C^{-1}(fdx + gdy) = dh$ for some $h \in \mathcal{O}_X$. Thus $f^p x^{p-1}dx + g^p y^{p-1}dy = dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$. Now, we know $f^p x^{p-1} = \sum \lambda_{i,j} y^{ip} x^{jp+p-1} = \frac{\partial h}{\partial x}$, but this is ridiculous because we claim that this is the derivative of some h with respect to x. If you take a derivative of some polynomial in x with respect to x, no output can ever have x^{jp+p-1} in it.

The surjectivity of C^{-1} is more involved. See for example, [Kat70], [Car57] or [EV92], [BK05], and follows similar lines to the proof of the next lemma. The isomorphism of the higher C^{-1} is an application of the Künneth formula.

We also need the following lemma.

Lemma 23.4. [Har98, Lemma 3.3] *With notation as in Proposition 23.2, additionally let $B = \sum r_j E_j$ be an effective integral divisor supported on E such that each $0 \leq r_j \leq p-1$. It follows that the inclusion of complexes (of $\mathcal{O}_X^p$-modules)*

$$\Omega^{\bullet}_X(\log E) \hookrightarrow (\Omega^{\bullet}_X(\log E))(B) := (\Omega^{\bullet}_X(\log E)) \otimes_{\mathcal{O}_X} \mathcal{O}_X(B)$$

is a quasi-isomorphism.

Proof. First we explain the differential on $(\Omega^{\bullet}_X(\log E))(B)$ because the tensor product with B is as an $\mathcal{O}_X$-module, it is not so clear what the differential is. However, we simply restrict the differential from $i_*\Omega^{\bullet}_{X \setminus E}$ to $(\Omega^{\bullet}_X(\log E))(B)$.

Now, the question is local, so we assume that X is the spectrum of a local ring. Choose $t_1, \dots, t_d$ to be local parameters (which also form a p-basis), where the components E_i of E are defined by $t_1, \dots, t_r$ respectively. Consider the complexes:

$$\mathscr{K}_j^{\bullet} = \left[0 \to \bigoplus_{i=0}^{p-1} t_j^i \mathcal{O}_X^p \to \bigoplus_{i=0}^{p-1} (t_j^i \frac{dt_j}{t_j^{\varepsilon_j}})\mathcal{O}_X^p\right]$$

where the middle-map is the usual d and where $\varepsilon_j = 1$ if $j \leq r$ and is zero otherwise. Set

$$\mathscr{J}_j^{\bullet} = t_j^{-r_j}\mathscr{K}_j^{\bullet},$$

for $j \leq r$.

We certainly have inclusions $\mathscr{K}_j^{\bullet} \subseteq \mathscr{J}_j^{\bullet}$, we claim that these are actually quasi-isomorphisms. We work in a very specific case, that of $k[x,y]$ where $E = \div X$. We only look at $\mathscr{K}_1$, of course the general case is exactly the same. We have the inclusion of complexes:

$$\begin{tikzcd} \oplus_{i=0}^{p-1} x^i \mathcal{O}_X^p \arrow[r] \arrow[d, hook] & \bigoplus_{i=0}^{p-1} x^i \frac{dx}{x} \arrow[d, hook] \\ \oplus_{i=0}^{p-1} x^{i-r} \mathcal{O}_X^p \arrow[r] & \bigoplus_{i=0}^{p-1} x^{i-r-1} dx. \end{tikzcd}$$

One can easily verify that the cokernel and kernel of the two rows "line-up" because r is between 0 and $p-1$. Thus we have proved our claim.

Now, we claim that

$$\Omega_X^{\bullet}(\log E) = \mathscr{K}_1^{\bullet} \otimes_{\mathcal{O}_X^p} \mathscr{K}_2^{\bullet} \otimes \ldots \otimes_{\mathcal{O}_X^p} \mathscr{K}_d^{\bullet}.$$

We'll check this for $X = \operatorname{Spec} \mathbb{F}_p[x,y]$ and $E = 0$. Here $\mathscr{K}_1 = \left[\bigoplus_{i=0}^{p-1} x^i \mathcal{O}_X^p \to \bigoplus_{i=0}^{p-1} (x^i dx) \mathcal{O}_X^p\right]$, and likewise $\mathscr{K}_2 = \left[\bigoplus_{i=0}^{p-1} y^i \mathcal{O}_X^p \to \bigoplus_{i=0}^{p-1} (y^i dy) \mathcal{O}_X^p\right]$. Thus $\mathscr{K}_1^{\bullet} \otimes \mathscr{K}_2^{\bullet}$ is the complex associated to the double-complex

$$\mathscr{K}_1^1 \otimes_{\mathcal{O}_X^p} \mathscr{K}_2^0 \cong (dx)\mathcal{O}_X \qquad \mathscr{K}^1 \otimes_{\mathcal{O}_X^p} \mathscr{K}^2 \cong (dx \wedge dy)\mathcal{O}_X$$

$$\mathscr{K}_1^0 \otimes_{\mathcal{O}_X^p} \mathscr{K}_2^0 \cong \mathcal{O}_X ar[u] \longrightarrow \mathscr{K}_1^0 \otimes_{\mathcal{O}_X^p} \mathscr{K}_2^1 \cong (dy)\mathcal{O}_X$$

The general case is similar, but messy to write down.

Arguing similarly, we have that

$$\Omega_X^{\bullet}(\log E)(B) \cong \mathscr{J}_1^{\bullet} \otimes \ldots \mathscr{J}_r^{\bullet} \otimes \mathscr{K}_{r+1}^{\bullet} \otimes \ldots \mathscr{K}_d^{\bullet}$$

and we have the natural (compatible) inclusion $\Omega_X^{\bullet}(\log E) \to \Omega_X^{\bullet}(\log E)(B)$ which are quasi-isomorphisms by the Künneth formula. $\square$

Now consider the following setup:

Let D be a $\mathbb{Q}$-divisor such that $\operatorname{Supp}(\{D\}) \subseteq \operatorname{Supp}(E)$. Set $B = -p\lfloor -D \rfloor + \lfloor -pD \rfloor = p\lceil D \rceil - \lceil pD \rceil$ and note it is an effective divisor supported in E whose coefficients are between 0 and $p-1$. Therefore, $(p-1)E - B$ is also such a divisor. Thus we have a quasi-isomorphism:

$$F_* \Omega_X^{\bullet}(\log E) \subseteq F_*(\Omega_X^{\bullet}(\log E)((p-1)E - B)).$$

Therefore, composition with C^{-1} gives us an isomorphism

$$\Omega_X^i(\log E) \cong \mathcal{H}^i\left(F_*(\Omega_X^{\bullet}(\log E)((p-1)E - B))\right).$$

Twisting by $\mathcal{O}_X(-E + \lceil D \rceil)$, we get an isomorphism

$$\begin{aligned} &\Omega_X^i(\log E)(-E + \lceil D \rceil) \\ \cong\ &\mathcal{H}^i\left(F_*(\Omega_X^{\bullet}(\log E)((p-1)E - B - pE + p\lceil D \rceil))\right) \\ \cong\ &\mathcal{H}^i\left(F_*(\Omega_X^{\bullet}(\log E)(-E + \lceil pD \rceil))\right). \end{aligned}$$

We denote the ith cocycle and coboundary of $F_*(\Omega_X^{\bullet}(\log E)(-E + \lceil pD \rceil)$ by $\mathcal{Z}^i$ and $\mathcal{B}^i$ respectively. Thus we have the following sequences for all i.

$$0 \to \mathcal{Z}^i \to F_*(\Omega_X^i(\log E)(-E + \lceil pD \rceil)) \to \mathcal{B}^{i+1} \to 0$$
$$0 \to \mathcal{B}^i \to \mathcal{Z}^i \to \Omega_X^i(\log E)(-E + \lceil D \rceil) \to 0$$

The second sequence, for $i = d$, is simply

(6)
$$0 \to \mathcal{B}^d \to \mathcal{Z}^d = F_*(\Omega_X^d(\log E)(-E + \lceil pD \rceil)) = F_*\omega_X(\lceil pD \rceil) \to \omega_X(\lceil D \rceil) \to 0.$$

Now assume

(a) $H^j(X, \Omega_X^i(logE)(-E + \lceil D \rceil)) = 0$ for $i + j = d + 1$ and $j > 1$.
(b) $H^j(X, \Omega_X^i(logE)(-E + \lceil pD \rceil)) = 0$ for $i + j = d$ and $j > 0$.

We will prove that

$$\begin{array}{cl} & H^0(X, F_*\omega_X(\lceil pD \rceil)) \\ = & \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X(\lfloor -pD \rfloor), \omega_X) \\ \to & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(\lfloor -D \rfloor), \omega_X) \\ = & H^0(X, \omega_X(\lceil D \rceil)) \end{array}$$

surjects.

Proof. Therefore, to show that we have our desired surjectivity, it is sufficient to show that $H^1(X, \mathcal{B}^d) = 0$. Thus, by the first short exact sequence, to show this, it is sufficient to show that $H^2(X, \mathcal{Z}^{d-1}) = 0$ and $H^1(X, F_*(\Omega_X^{d-1}(\log E)(-E + \lceil pD \rceil))) = 0$. The second of these is zero by hypothesis.

To show that $H^2(X, \mathcal{Z}^{d-1}) = 0$, by the second short exact sequence, it is sufficient to show that $0 = H^2(X, \mathcal{B}^{d-1}) = H^2(X, \Omega_X^{d-1}(\log E)(-E + \lceil D \rceil))$. The second of these is zero by hypothesis. Continuing in this way, to show that $H^2(X, \mathcal{B}^{d-1}) = 0$, it is sufficient to show that $H^3(X, \mathcal{Z}^{d-2}) = 0$, for which it is sufficient to show that $H^3(X, \mathcal{B}^{d-2}) = 0$, which eventually vanishes at $H^{d+1}(X, \mathcal{Z}^0) = 0$. $\square$

Now, all we have to show is that our desired vanishings (a), (b) actually hold (for $p \gg 0$). For D ample (b) should hold by Serre-vanishing for p large and (a) should hold by Kodaira-Akizuki-Nakano:

Theorem 23.5. [DI87], [Har98] *Suppose that X is d-dimensional and projective over a Noetherian affine scheme, and let D be an ample $\mathbb{Q}$-divisor with* $\mathrm{Supp}(\{D\}) \subseteq \mathrm{Supp}(E)$ *(where E is as before, a SNC divisor). Assume that $E \subseteq X$ admits a lifting to $W_2(k)$.*[11] *Then if $i + j > d$ and $p > d$, then*

$$H^j(X, \Omega_X^i(\log E)(-E + \lceil D \rceil)) = 0.$$

[11] This means there exists a smooth scheme $\widetilde{X}$ and a SNC divisor $\widetilde{E} = \sum_i \widetilde{E}_i$ over $\mathrm{Spec}\, W_2(k)$ with $\widetilde{X} = X \times_k W_2(k)$ and $\widetilde{E}_i = E_i \times_k W_2(k)$.

Proof. The result will be a corollary of the following result of Deligne-Illusie, with notation as above we have a quasi-isomorphism of $\mathcal{O}_X$-modules:

$$\bigoplus_{i=0}^{d} \Omega^i_X(\log E)[-i] \cong F_* \Omega^{\bullet}_X(\log E).$$

To see this, notice that we already had a quasi-isomorphism

$$F_*\Omega^{\bullet}_X(\log E) \cong F_*(\Omega^{\bullet}_X(\log E))((p-1)E - B))).$$

Twisting by $\mathcal{O}_X(-E + \lceil D \rceil)$ gives us a quasi-isomorphism

$$\bigoplus_{i=0}^{d} \Omega^i_X(\log E)(-E + \lceil D \rceil)[-i] \cong F_*\Omega^{\bullet}_X(\log E)(-E + \lceil pD \rceil).$$

Taking (hyper-)cohomology, we get

$$\oplus_{i+j=m} H^j(X, \Omega^i_X(\log E)(-E + \lceil D \rceil)) \cong \mathbb{H}^m(X, \Omega^{\bullet}_X(\log E)(-E + \lceil pD \rceil)).$$

Remember, we are trying to show that the terms of the left side are zero for $i + j = m > d$. But we also have the Hodge-to-De Rham spectral sequence

$$E_1^{ji} := H^j(X, \Omega^i_X(\log E)(-E + \lceil pD \rceil) \Rightarrow \mathbb{H}^m(X, \Omega^{\bullet}_X(\log E)(-E + \lceil pD \rceil))$$

and so it suffices to show that the terms $H^j(X, \Omega^i_X(\log E)(-E + \lceil pD \rceil)$ vanish for $i + j > d$. Repeating this process, it suffices to show that the terms

$$H^j(X, \Omega^i_X(\log E)(-E + \lceil p^e D \rceil)$$

vanish for $i + j > d$ and $e \gg 0$. But this is obvious by Serre vanishing. $\square$

We now do the following reduction to characteristic $p \gg 0$ statement.

Lemma 23.6. [Har98] *Begin with X, E, D as before, but in characteristic zero. The following vanishings hold for reduction to characteristic $p \gg 0$.*

(a) $H^j(X_p, \Omega^i_{X_p}(\log E_p)(-E_p + \lceil p^e D_p \rceil)) = 0$ *for* $i + j > d$ *and* $e \geq 0$.

(b) $H^j(X_p, \Omega^i_{X_p}(\log E_p)(-E_p + \lceil p^{e+1} D_p \rceil) = 0$ *for* $j > 0$ *and* $e \geq 0$.

Proof. The reason that these do not follow from standard reduction to characteristic p is because the twisting p involved depends on the actual sheaf in question. We need uniform vanishing results! Suppose A is the finitely generated $\mathbb{Z}$-algebra over which we do the reduction mod p (ie, $X_A \otimes_A \mathbb{C} = X$ and $X_A \otimes_A A/\mathfrak{p} = X_p$ for some maximal ideal $\mathfrak{p} \in \operatorname{Spec} A$).

Consider the quasi-coherent sheaf

$$\mathscr{F}_A = \bigoplus_{n \geq 0} \Omega^i_{X_A/A}(\log E_A)(-E_A + \lceil nD_A \rceil).$$

For each j, $H^j(X_A, \mathscr{F}_A)$ is a finitely generated module of $\mathcal{R}(X_A, D_A) := \oplus H^0(X_A, \mathcal{O}_{X_A}(\lfloor nD_A \rfloor))$ which itself is a finitely generated A-algebra (remember, D_A is ample). So by generic freeness, we may assume that $\mathscr{F}_A$ is (locally) A-free, and thus each graded piece $\Omega^i_{X_A/A}(\log E_A)(-E_A + \lceil nD_A \rceil)$ is also (locally) A-free.

Therefore,

$$H^j(X_A, \Omega^i_{X_A/A}(\log E_A)(-E_A + \lceil nD_A \rceil)) \otimes_A A/bp = H^j(X_p, \Omega^i_{X_p}(\log E_p)(-E_p + \lceil nD_p \rceil)).$$

In particular, if the given vanishing (for a fixed n) holds for some $\mathfrak{p}$, they hold for all maximal $\mathfrak{p} \in \operatorname{Spec} A$. To prove (a), we'd need to show that the required lifting properties are satisfied, for some $\mathfrak{p}$. But for a sufficiently general $\mathfrak{p}$, the lifting properties required are satisfied!

For condition (b), we know that there exists an $n_0 \geq 0$ such that

$$H^j(X_A, \Omega^i_{X_A/A}(\log E_A)(-E_A + \lceil nD_A \rceil)) = 0$$

for some $j > 0$ and all $n \geq n_0$. But then since the characteristic of $A/\mathfrak{p} \geq n_0$ for a Zariski-dense set of $\mathfrak{p} \in \operatorname{Spec} A$, we are done. $\square$

24. Globally F-regular varieties

Definition 24.1. Let (X, Δ) be a pair, where X is a normal irreducible F-finite scheme of prime characteristic p and Δ is an effective $\mathbb{Q}$ divisor on X. The pair (X, Δ) is *globally F-regular* if, for every effective divisor D, there exists some $e > 0$ such that the natural map $\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$ splits (in the category of $\mathcal{O}_X$-modules).

X itself is called *globally F-regular* if $(X, 0)$ is globally F-regular.

Lemma 24.2. *If (X, Δ) is globally F-regular, then (X, Δ') is globally F-regular for any $\Delta' \leq \Delta$. The corresponding statement for globally sharply F-split pairs also holds.*

Proof. This follows easily from the following simple observation: If a map of coherent sheaves $\mathscr{L} \xrightarrow{g} \mathscr{F}$ on a scheme X splits, then there is also a splitting for any map $\mathscr{L} \xrightarrow{h} \mathscr{M}$ through which g factors. Indeed, factor g as $\mathscr{L} \xrightarrow{h} \mathscr{M} \xrightarrow{h'} \mathscr{F}$. Then if $s : \mathscr{F} \to \mathscr{L}$ splits g, it is clear that the composition $s \circ h'$ splits h. Now we simply observe that if $\Delta' \leq \Delta$, we have a factorization

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta' \rceil + D) \hookrightarrow F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D),$$

so the result follows. $\square$

Remark 24.3. On an affine variety, Globally F-regular is the same as strongly F-regular (one can certainly take $D = \operatorname{div}(c)$ for various $c \in \mathcal{O}_X$, and every effective divisor D is less than or equal to a Cartier divisor). However, since every globally F-regular variety is clearly F-split, not every (locally) strongly F-regular variety is globally F-regular.

We now establish a useful criterion for global F-regularity, generalizing well-known results for the local case [HH89, Theorem 3.3] and the "boundary-free" case [Smi00a, Theorem 3.10].

Theorem 24.4. *The pair (X, Δ) is globally F-regular if (and only if) there exists some effective (usually ample) divisor C on X satisfying the following two properties:*

(i) *There exists an $e > 0$ such that the natural map*

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta + C \rceil)$$

splits.

(ii) *The pair $(X \setminus C, \Delta|_{X \setminus C})$ globally F-regular (for example, affine and locally F-regular).*

Proof of Theorem 24.4. Let X_C denote the open set complimentary to C. Now fix any effective divisor C' on X. By hypothesis (ii), we can find e' and an $\mathcal{O}_X$-module homomorphism $\phi : F^{e'}_* \mathcal{O}_{X_C}(\lceil (p^{e'} - 1)\Delta|_{X_C} + C'|_{X_C} \rceil) \to \mathcal{O}_{X_C}$ that sends 1 to 1. In other words, ϕ is a section of the reflexive sheaf

$$\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^{e'}_* \mathcal{O}_X(\lceil (p^{e'} - 1)\Delta + C' \rceil), \mathcal{O}_X)$$

over the open set X_C. Thus on the non-singular locus U of X (really, we need the Cartier locus of C), we can choose $m_0 > 0$ so that $\phi|_U$ is the restriction of a global section ϕ_m of

$$\begin{aligned} &\mathscr{H}\mathrm{om}_{\mathcal{O}_U}(F^{e'}_* \mathcal{O}_U(\lceil (p^{e'} - 1)\Delta + C' \rceil), \mathcal{O}_U) \otimes \mathcal{O}_U(mC) \\ &\cong \mathscr{H}\mathrm{om}_{\mathcal{O}_U}(F^{e'}_* \mathcal{O}_U(\lceil (p^{e'} - 1)\Delta + C' \rceil), \mathcal{O}_U(mC)) \end{aligned} \tag{7}$$

over U, for all $m \geq m_0$; see [Har77, Chapter II, Lemma 5.14(b)]. Note that ϕ_m still sends 1 to 1. Now, since the involved sheaves are reflexive, this section extends uniquely to a global section of $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^{e'}_* \mathcal{O}_X(\lceil (p^{e'} - 1)\Delta + C' \rceil), \mathcal{O}_X(mC))$, also denoted ϕ_m over the whole of X.

Consider an m of the form $m = p^{(n-1)e} + \dots p^e + 1$, where e is the number guaranteed by hypothesis (i). Tensoring the map ϕ_m from Equation (7) with $\mathcal{O}_X(\lceil (p^{ne} - 1)\Delta \rceil)$, we have an induced map

$$F^{e'}_* \mathcal{O}_X(\lceil (p^{e'} - 1)\Delta \rceil + C' + p^{e'} \lceil (p^{ne} - 1)\Delta \rceil) \to \mathcal{O}_X(\lceil (p^{ne} - 1)\Delta + mC \rceil).$$

Now, as in Lemma 24.2, it follows that there is a map

$$\psi : F^{e'}_* \mathcal{O}_X(\lceil (p^{ne+e'} - 1)\Delta + C' \rceil) \to \mathcal{O}_X(\lceil (p^{ne} - 1)\Delta + mC \rceil)$$

which sends 1 to 1.

By composing the splitting from hypothesis (i) with itself $(n-1)$-times and after twisting appropriately (compare with [Tak04b, Proof of Lemma 2.5] and [Sch09]), we obtain a map

$$\theta : F^{ne}_* \mathcal{O}_X(\lceil (p^{ne} - 1)\Delta + (p^{(n-1)e} + \dots + p^e + 1)C \rceil) = F^{ne}_* \mathcal{O}_X(\lceil (p^{ne} - 1)\Delta + mC \rceil) \to \mathcal{O}_X$$

which sends 1 to 1.

Combining the maps θ and ψ, we obtain a composition

$$F^{ne+e'}_* \mathcal{O}_X(\lceil (p^{ne+e'} - 1)\Delta + C' \rceil) \xrightarrow{F^{ne}_*(\psi_\Delta)} F^{ne}_* \mathcal{O}_X(\lceil (p^{ne} - 1)\Delta + mC \rceil) \xrightarrow{\theta} \mathcal{O}_X$$

which sends 1 to 1 as desired. The proof is complete. $\square$

Theorem 24.5. *Let X be a normal scheme quasi-projective over an F-finite local ring with a dualizing complex and suppose that B is an effective $\mathbb{Q}$-divisor on X.*

(i) *If the pair (X, B) is globally F-regular, then there is an effective $\mathbb{Q}$-divisor Δ such that $(X, B+\Delta)$ is globally F-regular with $K_X + B + \Delta$ anti-ample.*
(ii) *Similarly, if (X, B) is globally sharply F-split, then there exists an effective $\mathbb{Q}$-divisor Δ such that $(X, B+\Delta)$ is globally sharply F-split with $K_X + B + \Delta$ $\mathbb{Q}$-trivial.*

In both (i) and (ii), the denominators of the coefficients of $B+\Delta$ can be assumed not divisible by the characteristic p.

Proof of Theorem 24.5. First, without loss of generality, we may assume that the $\mathbb{Q}$-divisor B has no denominators divisible by p, we won't prove this here but it is straightforward.

We first prove statement (ii), which follows quite easily. Suppose that (X, B) is globally sharply F-split. Consider a splitting

$$\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X((p^e-1)B) \xrightarrow{\phi} \mathcal{O}_X$$

where $(p^e-1)B$ is an integral divisor. Apply $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\_\_, \mathcal{O}_X)$ to this splitting. We then obtain the following splitting,

$$\mathcal{O}_X \longleftarrow F^e_* \mathcal{O}_X((1-p^e)K_X) \longleftarrow F^e_* \mathcal{O}_X((1-p^e)(K_X+B)) \xleftarrow{\phi^\vee} \mathcal{O}_X.$$

The image of 1 under $\phi^\vee$ determines a divisor D' which is linearly equivalent to $(1-p^e)(K_X+B)$. This produces a composition

$$\mathcal{O}_X \longleftarrow F^e_* \mathcal{O}_X(D' + (p^e-1)B) \longleftarrow F^e_* \mathcal{O}_X(D') \xleftarrow{\phi^\vee} \mathcal{O}_X. \tag{8}$$

Set $\Delta_1 = \frac{1}{p^e-1}D'$. Then the pair $(X, B+\Delta_1)$ is globally sharply F-split with the splitting given by Equation (8). But also, it is log Calabi Yau, since

$$K_X + B + \Delta_1 \sim_{\mathbb{Q}} K_X + B + \frac{1}{p^e-1}(1-p^e)(K_X+B) = 0.$$

This completes the proof of (ii).

More work is required to prove (i). Suppose that (X, B) is globally F-regular. Then it is also globally sharply F-split, and we may pick Δ_1 as in (ii). Choose H to be a very ample effective divisor such that $\operatorname{Supp}\Delta_1 \subseteq \operatorname{Supp} H$. Consider a splitting

$$\mathcal{O}_X \longrightarrow F^f_* \mathcal{O}_X(H) \longrightarrow F^f_* \mathcal{O}_X((p^f-1)B + H) \xrightarrow{\psi} \mathcal{O}_X,$$

such that $(p^f-1)B$ is integral. Apply $\mathscr{H}om_{\mathcal{O}_X}(\_\_, \mathcal{O}_X)$ to this splitting to obtain a dual splitting,

(9)

$$\mathcal{O}_X \longleftarrow F^f_* \mathcal{O}_X((1-p^f)K_X - H) \longleftarrow F^f_* \mathcal{O}_X((1-p^f)(K_X+B) - H) \xleftarrow{\psi^\vee} \mathcal{O}_X$$

The image of 1 under $\psi^\vee$ determines a divisor D'' which is linearly equivalent to $(1-p^f)(K_X+B)-H$. Set $\Delta_2 = \frac{1}{p^f-1}D''$. Note that

$$K_X + B + \Delta_2 \sim_{\mathbb{Q}} \frac{-1}{p^f-1}H$$

which is anti-ample. Also note that the splitting in line (9) demonstrates the pair $(X, B+\Delta_2)$ to be globally sharply F-split. Even better, line (9) also demonstrates $(X, B+\Delta_2+\frac{1}{p^f-1}H)$ to be globally sharply F-split.

We now make use of Lemma 24.6 below to complete the proof. In addition to the globally F-regular pair (X,B), we have constructed divisors Δ_1 and Δ_2 satisfying

(i) $(X, B+\Delta_1)$ is globally sharply F-split with $K_X+B+\Delta_1 \sim_{\mathbb{Q}} 0$; and
(ii) $(X, B+\Delta_2)$ is globally sharply F-split with $K_X+B+\Delta_2$ anti-ample.
(iii) $(X, B+\Delta_2+\delta H)$ is globally sharply F-split for some small positive δ.

Now we apply Lemma 24.6(i) to the divisors described in (i) and (iii) above. We thus fix positive rational numbers ϵ_1, ϵ_2, with $\epsilon_1+\epsilon_2=1$ such that

$$(X, \epsilon_1(B+\Delta_1)+\epsilon_2(B+\Delta_2+\delta H)) = (X, B+\epsilon_2\Delta_2+\epsilon_1\Delta_1+\epsilon_2\delta H)$$

is globally sharply F-split. Since the support of Δ_1 is contained in the support of H, it follows from Lemma 24.2 that

$$(X, B+\epsilon_2\Delta_2+(\epsilon_1+\epsilon')\Delta_1) \tag{10}$$

is globally sharply F-split for some small positive ϵ'. But also $(X, B+\epsilon_2\Delta_2)$ is globally F-regular, as one sees by applying Lemma 24.6(iii) to the globally F-regular pair (X,B) and the globally sharply F-split pair $(X, B+\Delta_2)$.

Finally, another application of Lemma 24.6(iii), this time to the globally F-regular pair $(X, B+\epsilon_2\Delta_2)$ and the globally sharply F-split pair of line (10), implies that $(X, B+\epsilon_2\Delta_2+\epsilon_1\Delta_1)$ is globally F-regular. Set $\Delta = \epsilon_1\Delta_1+\epsilon_2\Delta_2$. We conclude that the pair $(X, B+\Delta)$ is globally F-regular, and

$$K_X+B+\Delta = \epsilon_1(K_X+B+\Delta_1)+\epsilon_2(K_X+B+\Delta_2)$$

is anti-ample (from (i) and (ii) just above). This completes the proof of (i) and hence Theorem 24.5. $\square$

Lemma 24.6. *Consider two pairs (X,B) and (X,D) on a normal F-finite scheme X.*

(i) *If both pairs are globally sharply F-split, then there exist positive rational numbers ϵ arbitrarily close to zero such that the pair $(X, \epsilon B+(1-\epsilon)D)$ is globally sharply F-split.*

(ii) *If (X, B) is globally F-regular and (X, D) is globally sharply F-split, then there exist positive rational numbers ϵ arbitrarily close to zero such that the pair $(X, \epsilon B + (1-\epsilon)D)$ is globally F-regular.*

(iii) *In particular, if (X, B) is globally F-regular and $(X, B+\Delta)$ is globally sharply F-split, then $(X, B+\delta\Delta)$ is globally F-regular for all rational $0 < \delta < 1$.*

In (i) and (ii), the number ϵ can be assumed to have denominator not divisible by p.

Proof of Lemma 24.6. First note that (iii) follows from (ii) by taking D to be $(B+\Delta)$. Since $(1-\epsilon)$ can be taken to be arbitrarily close to 1, we can choose it to exceed any given $\delta < 1$. Hence, the pair $(X, B+\delta\Delta)$ is globally F-regular for all positive $\delta < 1$, by Lemma 24.2.

For (i), we prove that we can take ϵ to be any rational number of the form

$$\epsilon = \frac{p^e - 1}{p^{(e+f)} - 1} \tag{11}$$

where e and f are sufficiently large and divisible (but independent) integers. Take e large and divisible enough so there exists a map $\phi : F^e_* \mathcal{O}_X(\lceil (p^e - 1)B \rceil) \to \mathcal{O}_X$ which splits the map $\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)B \rceil)$. Likewise, take f large and divisible enough so there exists a map $\psi : F^f_* \mathcal{O}_X(\lceil (p^f - 1)D \rceil) \to \mathcal{O}_X$ which splits the map $\mathcal{O}_X \to F^f_* \mathcal{O}_X(\lceil (p^f - 1)D \rceil)$.

Consider the splitting

$$\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(\lceil (p^e - 1)B \rceil) \xrightarrow{\phi} \mathcal{O}_X.$$

Because all the sheaves above are reflexive and X is normal, we can tensor with $\mathcal{O}_X(\lceil (p^f - 1)D \rceil$ to obtain a splitting

$$\mathcal{O}_X(\lceil (p^f - 1)D \rceil) \longrightarrow F^e_* \mathcal{O}_X(\lceil (p^e - 1)B \rceil + p^e \lceil (p^f - 1)D \rceil) \longrightarrow \mathcal{O}_X(\lceil (p^f - 1)D \rceil).$$

Applying F^f_* to this splitting, and then composing with ψ we obtain the following splitting,

$$\mathcal{O}_X \longrightarrow F^{e+f}_* \mathcal{O}_X(\lceil (p^e - 1)B \rceil + p^e \lceil (p^f - 1)D \rceil) \longrightarrow \mathcal{O}_X$$

However, we also note that

$$\lceil (p^e - 1)B \rceil + p^e \lceil (p^f - 1)D \rceil \geq \lceil (p^e - 1)B + p^e (p^f - 1)D \rceil$$

which implies that we also have a splitting

$$\mathcal{O}_X \longrightarrow F^{e+f}_* \mathcal{O}_X(\lceil (p^e - 1)B + p^e (p^f - 1)D \rceil) \longrightarrow \mathcal{O}_X$$

If we then multiply $(p^e - 1)B + p^e(p^f - 1)D$ by $\frac{1}{p^{(e+f)}-1}$, the proof of (i) is complete for the choice of ϵ given in line 11.

Now, to prove (ii), we use Theorem 24.4. Choose an effective integral divisor C whose support contains the support of D and such that the pair $(X \setminus C, D|_{X\setminus C})$ is globally F-regular. Since there exists a splitting of

$$\mathcal{O}_X \to F^f_* \mathcal{O}_X(\lceil (p^f - 1)B + C \rceil),$$

it follows that the pair $(X, B + \frac{1}{p^f-1}C)$ is globally sharply F-split. Applying part (i) of the Lemma to the pairs $(X, B + \frac{1}{p^f-1}C)$ and (X, D), we conclude that

$$(X, \epsilon(B + \frac{1}{p^f - 1}C) + (1 - \epsilon)D)$$

is globally sharply F-split. Re-writing, we have

$$(X, \epsilon B + (1 - \epsilon)D + \epsilon' C)$$

is globally sharply F-split for ϵ and ϵ' arbitrarily close to zero.

We now apply Theorem 24.4 to the pair $(X, \Delta) = (X, \epsilon B + (1 - \epsilon)D)$. Restricted to $X \setminus C$, this pair is globally F-regular, and we've just shown that for sufficiently small ϵ', the pair $(X, \Delta + \epsilon' C)$ is globally sharply F-split. Using Lemma 24.4 we conclude that (X, Δ) is globally F-regular.

Finally, note that because of the explicit choice of ϵ in line (3), it is clear its denominator can be assumed not divisible by p. ☐

Corollary 24.7. *If X is globally F-regular, then X there exists a divisor $\Delta \geq 0$ such that (X, Δ) is log Fano.*

Straightforward techniques involving cones imply the following converse.

Theorem 24.8. *Let X be a normal projective variety over a field of characteristic zero. If (X, Δ) is a Kawamata log terminal pair such that $K_X + \Delta$ is anti-ample (ie, (X, Δ) is log Fano), then (X, Δ) has globally F-regular type.*

Proof. The idea of the proof is the following lemma. X in characteristic $p > 0$ is globally F-regular if and only if the section ring with respect to an ample divisor is strongly F-regular. Also, for X in characteristic zero, (X, Δ) is log Fano if and only if the section ring pair (S, Δ_S), associated to an ample divisor, is Kawamata log terminal. Now reduce to characteristic $p \gg 0$. ☐

Theorem 24.9. *Let X be a normal projective variety over a field of prime characteristic. Let L be a Cartier divisor on X such that $L \sim_{\mathbb{Q}} M + \Delta$, where M is a nef and big $\mathbb{Q}$-divisor and the pair (X, Δ) is globally F-regular. Then $H^i(X, \mathcal{O}_X(-L)) = 0$ for $i < \dim X$.*

Proof. Because L is big, we can fix $f \gg 0$ so that there exists an effective E linearly equivalent to $p^f L$. By taking f larger if necessary, we can also assume that for all large and sufficiently divisible e,

(1) $p^f(p^e - 1)\Delta$ and $p^f(p^e - 1)M$ are integral,
(2) $\mathcal{O}_X(p^f(p^e - 1)L) \cong \mathcal{O}_X(p^f(p^e - 1)(M + \Delta))$.

Since M is nef and big, there exists an effective divisor D such that $nM - D$ is ample for all $n \gg 0$; see [Laz04, Cor 2.2.7]. Because (X, Δ) is globally F-regular, for all sufficiently large integers g, the map

$$\mathcal{O}_X \to F^g_* \mathcal{O}_X(\lceil (p^g - 1)\Delta \rceil + D + E)$$

splits. By choosing g large enough, we may assume that $g = f + e$ where f is the fixed integer above and $e > 0$ is such that both (1) and (2) are satisfied above. Also, we can assume that $p^f(p^e - 1)M - D$ is ample. Therefore, the map

$$\mathcal{O}_X \to F^{e+f}_* \mathcal{O}_X(p^f(p^e - 1)\Delta + D + E)$$

splits since $p^f(p^e - 1)\Delta \leq \lceil (p^{e+f} - 1)\Delta \rceil$. Tensoring (on the smooth locus, and extending as usual) with $\mathcal{O}_X(-L)$ and taking cohomology, we have a splitting of the map

$$H^i(X, \mathcal{O}_X(-L)) \to H^i(X, F^{e+f}_* \mathcal{O}_X(-p^{e+f}L + p^f(p^e - 1)\Delta + D + E)).$$

In particular, this map on cohomology is injective for all sufficiently large and divisible e.

However,

$$\begin{aligned} -p^{e+f}L + p^f(p^e - 1)\Delta + D + E = \\ -(p^{e+f} - p^f)L - p^f L + p^f(p^e - 1)\Delta + D + E \sim \\ (-p^f(p^e - 1)M - p^f(p^e - 1)\Delta) + p^f(p^e - 1)\Delta + D + \big(E - p^f L\big) \sim \\ -p^f(p^e - 1)M + D \end{aligned}$$

which is anti-ample. Therefore, $H^i(X, \mathcal{O}_X(-p^{e+f}L + p^f(p^e - 1)\Delta + D + E))$ vanishes for $i < \dim X$ since X is globally F-regular, by [Smi00a, Corollary 4.4], see also [BK05]. Because of the injection above, it follows that $H^i(X, \mathcal{O}_X(-L))$ vanishes, and the proof is complete. $\square$

25. Criteria for F-splitting of varieties

In the past, we've see Fedder's criteria for Frobenius splitting of algebraic varieties. Now, suppose that X is a variety over an algebraically closed field of characteristic $p > 0$. We will discuss the Mehta-Ramanathan criterion of Frobenius splitting, which is very useful in practice.

We've recently discussed using Cartier-operator as a way to construct explicitly the dual of Frobenius, $F_* \omega_X \to \omega_X$. Recall this was constructed as follows: we have the isomorphism $C^{-1} : \Omega^i_X(\log E) \cong \mathcal{H}^i\left(F_*(\Omega^\bullet_X(\log E)\right)$. Take $E = 0$ and $i = d = \dim X$, this give us $\omega_X \cong \mathcal{H}^d\left(F_* \Omega^\bullet_X\right)$. But for $i > d$, the terms $F_* \Omega^i_X$ of the complex $F_* \Omega^\bullet_X$ are zero, and so we have a surjection $F_* \omega_X \to \omega_X$. This can be identified with the canonical dual of Frobenius.

Lemma 25.1. [BK05, Lemma 1.3.6] *Suppose that $x \in X$ is a smooth point of an n-dimensional variety X over an algebraically closed field k. Then the* *map $T : F_*\omega_X \to \omega_X$ is described by the following formula. For any set of generators $t_1, \dots, t_n$ of the maximal ideal of $\mathcal{O}_{X,x}$*

$$T(f dt_1 \wedge \cdots \wedge dt_n) = S(f) dt_1 \wedge \cdots \wedge dt_n$$

where S is defined on $k[[t_1, \dots, t_n]] \supseteq \mathcal{O}_{X,x}$ as the map which sends the monomial $t_1^{p-1} \dots t_n^{p-1}$ to 1 and the other monomials to zero.

This proof is taken from [BK05]. Certainly $dt_1 \wedge \dots dt_n$ generates ω_X as an $\mathcal{O}_X$-module as well, which identifies $\omega_{X,x}$ with $\mathcal{O}_{X,x}$. The completion of $\omega_X / d(\Omega_X^{d-1})$ is thus identified with $k[[x_1, \dots, x_n]]/J$ where J is the *vector-space* spanned by all partial derivatives of $h \in k[[x_1, \dots, x_n]$. To see this, simply note that

$$d(h\widehat{dt_i}) = \partial h \partial t_i dt_1 \wedge \cdots \wedge dt_n.$$

Thus, J is made up of all power series $\sum a_{\mathbf{i}} t^{\mathbf{i}}$ where $p \not| (\mathbf{i}_j + 1)$ for some $1 \leq j \leq n$. In other words, $k[[x_1, \dots, x_n]]/J$ is the set of power-series of the form $\sum a_{\mathbf{j}} t^{\mathbf{p-1}+p\mathbf{j}}$. But this is obviously identified with $(k[[t_1, \dots, t_n]])^p$, and unraveling our identifications yields the desired formula. $\square$

Following Brion and Kumar, we also obtain the following:

Proposition 25.2. [BK05, Proposition 1.3.7] *Let X be a nonsingular variety. Then the following map η is an isomorphism. The map η*

$$\eta : \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \to \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

is defined as follows: Working locally, fix a local generator ω for $\omega_{X,x}$. Furthermore, for $\psi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_{X,x}}(\omega_{X,x}, F_*\omega_{X,x})$ and $f \in \mathcal{O}_{X,x}$, we define $\eta(\psi)f$ to be the ω coefficient of $T(f\psi(\omega))$.

This is well defined and furthermore, we obtain the following commutative diagram

$$\begin{CD}
\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X, F_*\omega_X) @>{\eta}>> \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \\
@V{T}VV @VV{\text{eval at 1}}V \\
\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X, \omega_X) @>{\kappa}>> \mathscr{H}\mathrm{om}(\mathcal{O}_X, \mathcal{O}_X)
\end{CD}$$

where κ is the natural isomorphism.

Proof. Fix $g \in \mathcal{O}_{X,x}$. Then notice that $\eta(\psi \cdot g)$ is defined by the rule

$$T(f\psi(g\omega))/\omega = T(fg^p\psi(\omega))/\omega = gT(f\psi(\omega))/\omega$$

In particular, η is $F_*\mathcal{O}_X$-linear.

We now show that our local definition of η is well defined. Suppose that $\omega' = u\omega$ for some unit $u \in \mathcal{O}_{X,x}$. With this, we define a new map η', where $\eta'(\psi)(f) = T(f\psi(\omega'))/\omega'$. So,

$$\eta'(\psi)(f) = T(f\psi(\omega'))/\omega' = T(f\psi(u\omega))/\omega' = T(fu^p\psi(\omega))/\omega' = uT(f\psi(\omega))/(u\omega) = \eta(\psi)(f).$$

Now we show that the diagram commutes. Given $\psi \in \mathscr{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X)$, the left-vertical arrow is defined by:

$$(T(\psi))(f\omega) = T(\psi(f\omega)) = fT(\psi(\omega)).$$

In particular, $\kappa(T(\psi))$ is the map obtained by multiplication by $T(\psi(\omega))/\omega$. On the other hand, the composition of η with the right vertical arrow is just

$$\eta(\psi)(1) = T(\psi(\omega))/\omega.$$

Therefore, the diagram commutes as desired.

Finally, we show that η is an isomorphism. We work locally and fix a minimal set of generators $x_1, \dots, x_n$ for the maximal ideal of $\mathcal{O}_{X,x}$. Notice that $\psi \in \mathscr{H}om_{\mathcal{O}_{X,x}}(\omega_{X,x}, F_*\omega_{X,x})$, defined by the rule

$$\psi(f dt_1 \wedge \cdots \wedge dt_n) = f^p dt_1 \wedge \cdots \wedge dt_n$$

This map clearly generates $\mathscr{H}om_{\mathcal{O}_X}(\omega_{X,x}, F_*\omega_{X,x})$ as an $F_*\mathcal{O}_{X,x}$-module. Now, $\eta(\psi)(f) = T(f\psi(\omega))/\omega = T(f\omega) = S(f)$. In particular, since S generates $\mathrm{Hom}_{\mathcal{O}_{X,x}}(F_*\mathcal{O}_{X,x}, \mathcal{O}_{X,x})$, we see that η is surjective, and thus it is an isomorphism since both modules are rank-1 $F_*\mathcal{O}_X$-modules. $\square$

Now, $\mathscr{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \cong F_* \mathscr{H}om_{\mathcal{O}_X}(F^*\omega_X, \omega_X) \cong F_*\omega_X^{1-p}$. This yields a *canonical* isomorphism:

$$\alpha : \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong F_*\omega_X^{(1-p)}$$

Theorem 25.3. [BK05, Theorem 1.3.8] [MR85] *The evaluation-at-1 map $\mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{O}_X$ is identified the map*

$$\sigma : F_*\omega_X^{(1-p)} \to \mathcal{O}_X$$

defined locally by

$$\sigma(f(dt_1 \wedge \dots dt_n)^{1-p}) = S(f).$$

Therefore, $\phi \in \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ splits the Frobenius map if and only if $\sigma(\alpha(\phi)) = 1$.

Proof. The diagram in the previous proposition proves exactly the first claim, and the second follows immediately. $\square$

Corollary 25.4. *For a smooth X, given $\phi \in \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, if ϕ is a splitting, then the $\mathbf{t}^{p-1}\omega$ coefficient of $\alpha(\phi)$ is equal to 1 for every point $x \in X$. For a general normal complete X, $\phi \in \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is a splitting if and only if the $\mathbf{t}^{p-1}\omega$ coefficient of $\alpha(\phi)$ is equal to 1 for some point $x \in X$*

Proof. The $\mathbf{t}^{p-1}(dt_1 \wedge \dots dt_n)^{1-p}$-coefficient of $\alpha(\phi)$ is the constant term of $\phi(1)$ in $\mathcal{O}_{X,x} \subseteq k[[t_1, \dots, t_n]]$. Thus if $\phi(1) = 1$, this is just 1. For the complete case, we know that $\phi(1)$ is an element of $k = H^0(X, \mathcal{O}_X)$, and so the $\mathbf{t}^{p-1}(dt_1 \wedge \dots dt_n)^{1-p}$-coefficient of $\alpha(\phi)$ is the only term that matters. $\square$

We now come to the main result of this section. An effective tool for determining if a given $\phi \in \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong F_*\omega_X^{(1-p)}$ is a splitting.

Theorem 25.5. [BK05], [MR85] *Suppose that X is a normal complete variety of dimension n. If there exists a $s \in H^0(X, \omega_X^{(-1)})$ with associated divisor*

$$D = Y_1 + \dots Y_n + Z$$

where $Y_1, \dots, Y_n$ are prime divisors which intersect with SNC at a smooth closed point $x \in X$ and Z is an effective divisor not containing x, then X is Frobenius split by a splitting corresponding to $s^{p-1} \in H^0(X, \omega_X^{(1-p)})$ up to a unit.

More generally, if $s \in H^0(X, \omega_X^{(1-p)})$ is such that the divisor of s is $(p-1)(Y_1 + \dots + Y_n) + Z$ where the Y_1 are SNC at a closed point $x \in X$ and Z does not contain X, then the same result holds.

Proof. At $x \in X$, suppose that each Y_i is given by the vanishing of some $t_i \in \mathcal{O}_{X,x}$. Then the power series expansion of s^{p-1} is simply $t_1^{p-1} \dots t_n^{p-1} g(dt_1 \wedge \dots \wedge dt_n)^{1-p}$ where g is a formal power series not vanishing at the origin. In particular, the section $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)$ corresponding to s^{p-1} sends 1 to a non-zero constant in k. Multiplying by the inverse of that constant gives us our desired result. $\square$

Remark 25.6. It should be noted that the ϕ constructed above is compatible with all the Y_i's and Z, since the Δ_ϕ is exactly $Y_1 + \dots + Y_n + Z = \frac{(p-1)}{(p-1)}(Y_1 + \dots + Y_n + Z)$.

Corollary 25.7. *Suppose that X is a complete n-dimensional variety in characteristic zero and Δ is a $\mathbb{Q}$-divisor in characteristic zero such that $\Delta = Y_1 + \dots Y_n + Z$ where the Y_i are prime divisors which intersect with SNC at a smooth closed point $x \in X$ and Z is an effective divisor not containing x. Further suppose that $K_X + \Delta \sim_{\mathbb{Q}} 0$, then (X, Δ) is log canonical.*

Proof. Reduce to characteristic $p \gg 0$, then (X_p, Δ_p) is F-split and thus locally F-pure. This implies that (X, Δ) is log canonical. $\square$

26. Diagonal splitting

Definition 26.1. [RR85] Suppose that X is a variety. We say that X is *diagonally split* if the diagonal D is compatibly Frobenius split in $X \times X$. Given an ample divisor on X, we say that X is *diagonally split along an ample effective divisor* A if there exists a Frobenius splitting $\phi : F_* \mathcal{O}_{X \times X} \to \mathcal{O}_{X \times X}$ that compatibly splits D and also factors through $F_* \mathcal{O}_{X \times X}(p_2^* A)$.

Proposition 26.2. *Suppose that X is a complete variety and suppose that $\mathscr{L}$ and $\mathscr{M}$ are line bundles on X. Consider the natural map*

$$m(\mathscr{L}, \mathscr{M}) : \Gamma(X, \mathscr{L}) \otimes \Gamma(X, \mathscr{M}) \to \Gamma(X, \mathscr{L} \otimes \mathscr{M}).$$

If either

(1) *$\mathscr{L}$ and $\mathscr{M}$ are ample and X is diagonally Frobenius split, or*

(2) *$\mathscr{L}$ and $\mathscr{M}$ are semi-ample (or simple nef?) and X is diagonally Frobenius split along an ample effective Cartier divisor A,*

then $m(\mathscr{L}, \mathscr{M})$ is surjective.

Proof. We begin by recasting $m(\mathscr{L}, \mathscr{M})$ as a different map. Now, $\Gamma(X, \mathscr{L}) \otimes \Gamma(X, \mathscr{M}) \cong \Gamma(X \times X, p_1^* \mathscr{L} \otimes p_2^* \mathscr{M})$, and furthermore, if $i : D \subseteq X \times X$ is the inclusion map, then $i^*(p_1^* \mathscr{L} \otimes p_2^* \mathscr{M}) = \mathscr{L} \otimes \mathscr{M}$. Therefore, it is sufficient to show that the restriction map

$$\Gamma(X \times X, p_1^* \mathscr{L} \otimes p_2^* \mathscr{M}) \to \Gamma(D, (p_1^* \mathscr{L} \otimes p_2^* \mathscr{M})|_D)$$

is surjective. In the first case, $p_1^* \mathscr{L} \otimes p_2^* \mathscr{M}$ is ample. Consider the following commutative diagram where ϕ is just the Frobenius splitting twisted by a line bundle:

$$\begin{tikzcd}
H^0(X \times X, \mathcal{O}_{X \times X}((p_1^* \mathscr{L} \otimes p_2^* \mathscr{M})^{p^e})) \arrow[r, "\phi"] \arrow[d, "\gamma"'] & H^0(X \times X, \mathcal{O}_{X \times X}(p_1^* \mathscr{L} \otimes p_2^* \mathscr{M})) \arrow[d, "\delta"] \\
H^0(D, \mathcal{O}_D((p_1^* \mathscr{L} \otimes p_2^* \mathscr{M})^{p^e})) \arrow[r, "\bar{\phi}"] & H^0(D, \mathcal{O}_D(p_1^* \mathscr{L} \otimes p_2^* \mathscr{M}))
\end{tikzcd}$$

By Serre vanishing, γ is surjective and $\bar{\phi}$ is also surjective because it is induced from a splitting. Thus δ is surjective as well and (1) is proven.

By composing Frobenius splittings along an both $p_1^* A$ and $p_2^* A$, we obtain a Frobenius splitting along an ample divisor $B = p_1^* A^n \otimes p_2^* A^m$ on $X \times X$ for some integers $n, m > 0$. Consider the restriction map

$$H^0(X \times X, \mathcal{O}_{X \times X}((p_1^* \mathscr{L}^r \otimes p_2^* \mathscr{M}^r)(B))) \to H^0(D, \mathcal{O}_D((p_1^* \mathscr{L}^r \otimes p_2^* \mathscr{M}^r)(B)))$$

for various integers r. The above argument shows that this map is surjective. Composing with the Frobenius splitting along B gives us a diagram

$$\begin{tikzcd}
H^0(X \times X, F^e_* \mathcal{O}_{X \times X}((p_1^* \mathscr{L}^r \otimes p_2^* \mathscr{M}^r)(B))) \arrow[r] \arrow[d] & H^0(D, F^e_* \mathcal{O}_D((p_1^* \mathscr{L}^r \otimes p_2^* \mathscr{M}^r)(B))) \arrow[d] \\
H^0(X \times X, \mathcal{O}_{X \times X}((p_1^* \mathscr{L}^r \otimes p_2^* \mathscr{M}^r))) \arrow[r] & H^0(D, F^e_* \mathcal{O}_D((p_1^* \mathscr{L}^r \otimes p_2^* \mathscr{M}^r)))
\end{tikzcd}$$

As before, the bottom row is surjective which completes the proof. $\square$

Corollary 26.3. *Suppose that X is a diagonally Frobenius split projective variety, then every ample divisor is very ample and induces a projectively normal embedding. Furthermore, if it is diagonally Frobenius split along an ample divisor, then the algebra of sections of a semi-ample divisor is generated in degree 1. Furthermore, every semi-ample divisor is globally generated.*

Corollary 26.4. *Suppose in addition that X is Cohen-Macaulay and diagonally Frobenius split along an ample divisor, then X is arithmetically Cohen-Macaulay with respect to any ample line bundle.*

Proof. Choose A an ample effective divisor, this divisor is very ample and induces a projectively normal embedding by assumption. Thus we only have to show that $H^i(X, \mathcal{O}_X(vA)) = 0$ for $1 \leq i \leq \dim(X) - 1$ and all $v \in \mathbb{Z}$. But since A is ample, these vanishings hold via the usual Frobenius splitting arguments for $v \neq 0$. Consider $v = 0$, suppose that M is an ample divisor along which X is Frobenius split. We have

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X \to F^e_* \mathcal{O}_X(M)$$

splits and thus $\mathcal{O}_X \to F^{ne} \mathcal{O}_X(mM)$ where we can make $m \gg 0$. But then $H^i(X, \mathcal{O}_X) \to H^i(X, F^e_* \mathcal{O}_X(mM))$ splits, and the right side vanishes for $m \gg 0$. ☐

Remark 26.5. Various generalizations can be made to splittings of $X \times X \times \cdots \times X$. Furthermore, these can be used to prove that various section rings $R(X, \mathscr{L})$ are Koszul.

27. Toric varieties

In this section we briefly discuss Frobenius splittings on toric varieties. There are numerous good introductions to toric varieties available, the canonical reference is probably still [Ful93] although also see [CLS].

Consider the torus $T \simeq (\mathbb{G}_m)^n = (\mathbb{A}^1 \setminus \{0\})^n$ where $k = \overline{k}$.

Definition 27.1. A *toric variety* is a normal variety X containing T as an open subset such that the natural action of T on itself by multiplication extends to an action on X.

Lemma 27.2. *A toric variety can be covered by Torus invariant affine open subsets. Each one of them is* $\operatorname{Spec} k[\mathbf{x}^{\lambda_1}, \ldots, \mathbf{x}^{\lambda_m}]$ *for some monomials* $\mathbf{x}^{\lambda_i}$*.*

Proof. We leave the first statement to the reader, as it is contained in any introductory text on toric varieties. For the second statement, notice that if $U = \operatorname{Spec} R$ is a torus invariant open affine subset, then if any polynomial $f = \sum a_i \mathbf{x}^i$ is in U, by using the torus action, it is clear that each monomial appearing in f is in R. The claimed statement follows. ☐

Now, the torus $T = \operatorname{Spec} k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has a very natural Frobenius splitting $\Phi_c : F_* \mathcal{O}_T \to \mathcal{O}_T$, namely the one defined as follows:

$$\Phi_c(\mathbf{x}^\lambda) = \begin{cases} \mathbf{x}^{\lambda/p} & \text{if each entry in } \lambda \text{ is divisible by } p. \\ 0 & \text{otherwise.} \end{cases}$$

This is called the canonical Frobenius splitting (also see [BK05, Section 4]).

Proposition 27.3. *If X is a toric variety, then Φ_c induces a Frobenius splitting $\Phi_c : F_* \mathcal{O}_X \to \mathcal{O}_X$.*

Proof. We can work on an open affine set $U = \operatorname{Spec} R = \operatorname{Spec} k[\mathbf{x}^{\lambda_1}, \ldots, \mathbf{x}^{\lambda_m}]$. Since R is normal, if $\mathbf{x}^\lambda \in R$, then if $\lambda/p \in \mathbb{Z}^n$, we clearly see that $\mathbf{x}^{\lambda/p} \in R$ as well. Since $\Phi_c(1) = 1$, we have explicitly seen our Frobenius splitting. ☐

Definition 27.4. If X is a toric variety and $Z \subseteq X$ an irreducible subvariety, then we say that Z is a *torus invariant subvariety* if it is invariant under the torus action.

Example 27.5. In the toric variety $\mathbb{A}^2$, the torus invariant subvarieties are the two coordinate axes and also the origin.

Lemma 27.6. *Suppose that X is a toric variety, then $Z \subseteq X$ is a torus invariant subvariety if and only if Z is Φ_c compatible.*

Proof. We can assume that $X = \operatorname{Spec} k[\mathbf{x}^{\lambda_1}, \dots, \mathbf{x}^{\lambda_m}]$. A torus invariant subvariety thus corresponds to a prime ideal generated by monomials, and it is clear that any such ideal is Φ_c-compatible. Thus we suppose that Q is a Φ_c compatible ideal (note that Φ_c is surjective, so Q is automatically radical and $\Phi_c(F_* Q) = Q$). Further suppose that Q is prime. We will show that Q generated by monomials.

Suppose that $\sum a_i \mathbf{x}^{\lambda_i} = g \in Q$. We simply need to show that each $\mathbf{x}^{\lambda_i} \in Q$. Consider $h = \Phi_c^e(\mathbf{x}^{(p^e-1)\lambda_i} g)$. Clearly this polynomial contains $\mathbf{x}^{\lambda_i}$ as an entry. Now, consider $\Phi_c^e(\mathbf{x}^{(p^e-1)\lambda_i}\mathbf{x}^{\lambda_j}) = \mathbf{x}^{((p^e-1)\lambda_i+\lambda_j)/p^e}$. But

$$((p^e-1)\lambda_i + \lambda_j)/p^e = \lambda_i + \frac{\lambda_j - \lambda_i}{p^e}.$$

This is not in $\mathbb{Z}^n$ for $e \gg 0$ if $j \neq i$. Therefore, for $e \gg 0$, $\Phi_c^e(\mathbf{x}^{(p^e-1)\lambda_i} g) = \mathbf{x}^{\lambda_i}$ which proves that Q is generated by monomials. $\square$

We now briefly review the theory of canonical divisors on toric varieties.

Lemma 27.7. *The anti-canonical divisor $-K_X$ in a toric variety X is equal to the sum of all the torus invariant divisors. It can also be identified with $X \setminus T$.*

Proof. See for example, [Ful93]. $\square$

Proposition 27.8. *The Frobenius splitting Φ_c above has associated divisor $-K_X$.*

Proof. Clearly the divisor $\Delta_{\Phi_c} \geq -K_X$ (since every torus invariant divisor is Φ_c-compatible). Therefore, we only have to observe that $\operatorname{Supp}(\Delta_{\Phi_c})$ is torus invariant.

However, on the torus T, Φ_c generates $\operatorname{Hom}_{\mathcal{O}_T}(F_*\mathcal{O}_T, \mathcal{O}_T)$ as an $\mathcal{O}_T$-module. $\square$

Proposition 27.9. *Projective toric varieties in characteristic zero are log Fano and in characteristic $p > 0$ are globally F-regular.*

Proof. Suppose that X is a projective toric variety in characteristic zero. Suppose that A is an ample effective torus invariant divisor (it is a general fact that $\operatorname{Supp}(A) = \operatorname{Supp}(-K_X)$. Choose rational $\varepsilon > 0$ such that $\Delta := -K_X - \varepsilon A > 0$. Choose a toric log resolution $\pi : \widetilde{X} \to X$ of (X, Δ), we know that

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) = K_{\widetilde{X}} - \pi^*(K_X - K_X - \varepsilon A) = K_{\widetilde{X}} + \varepsilon \pi^* A.$$

It is clear that $\mathrm{Supp}(\pi^* A) = \mathrm{Supp}(-K_{\widetilde{X}})$, thus all the coefficients of $K_{\widetilde{X}} + \varepsilon \pi^* A$ are strictly bigger than -1 (which are the coefficients of $K_{\widetilde{X}}$).

Now suppose that X is a projective toric variety in characteristic $p > 0$. Again choose A an ample effective torus invariant divisor. The section ring $S(A)$ is always a monomial algebra, and therefore it is always strongly F-regular (since it is a summand of $k[x_1, \ldots, x_n]$). The fact that the section ring is strongly F-regular is then easily seen to imply that X is globally F-regular. $\square$

An important open question in the study of toric varieties is the following (which I have seen attributed to Oda):

Question 27.10. Suppose that X is a smooth toric variety and $\mathscr{L}$ is an ample line bundle. Then does $\mathscr{L}$ induce a projectively normal embedding into some projective space? It is known that $\mathscr{L}$ is always very ample.

Several years ago, it was hoped that the Frobenius splitting methods including Frobenius splitting along diagonals, would be enough to prove this result. The most immediate problem is that the diagonal in $X \times X$ is not torus invariant, and therefore it is *NOT* compatibly Frobenius split by Φ_c (this caused some confusion in the past). However Sam Payne has analyzed exactly when there exists a Frobenius splitting of $X \times X$ which compatibly splits the diagonal (it's just not the toric one).

If Δ is a fan in a lattice N and M is the dual lattice, Payne defined

$$\mathbf{F}_X := \{u \in M | -1 \leq \langle u, v_\rho \rangle \leq 1 \text{ where } v_\rho \text{ is a primitive generator of a ray in } \Delta.\}$$

Theorem 27.11. [Pay09] *A toric variety $X = X(\Delta)$ is diagonally Frobenius split if and only if the interior of $\mathbf{F}_X$ contains the interior of every equivalence class of $(\frac{1}{p}M)/M$.*

28. Kodaira-type vanishing in characteristic $p > 0$

First we recall Kodaira's vanishing theorem.

Theorem 28.1. [Kod53] *Suppose that X is a smooth projective variety of dimension n, characteristic zero, and H is an ample divisor on V, then*

$$H^i(X, \mathcal{O}_X(-H)) = 0$$

for $i = 0, 1, \ldots, n-1$. Dually, $H^i(X, \omega_X(H)) = 0$ for $i > 0$ (this dual version is equivalent as long as the variety is Cohen-Macaulay, which holds for example for normal surfaces).

This was known previously for surfaces, [Zar95]. It fails in characteristic zero for arbitrarily singular varieties (although it holds for normal surfaces), see for example [AJ89].

This result is also false in characteristic $p > 0$. We begin with Mumford's example (which is singular).

Example 28.2. [Mum67, Example 6] Suppose that X_0 is a normal surface in characteristic $p > 0$ with an element $\alpha \in H^1(X_0, \mathcal{O}_{X_0})$ such that $F(\alpha) = 0$ (for example, $X = E \times \mathbb{P}^1$ where E is a supersingular elliptic curve).

Suppose that H_0 is an irreducible hyperplane section of X_0 and let $L_0 = \mathcal{O}_{X_0}(H_0)$. Choose a open covering U_i of X_0 that principalizes H_0 and represent α as $\{\alpha_{ij}\}$ in Čech cohomology and choose $g_i \in \Gamma(U_i, \mathcal{O}_{X_0})$ so that $\alpha_{ij}^p = g_i - g_j$. Suppose that $H_0|_{U_i} = V(h_i)$ for some $h_i \in \Gamma(U_i, \mathcal{O}_{X_0})$. Define an extension L of $K(X)$ by adjoining all roots of the equations:

$$z_i^p - h_i^p z_i = g_i$$

Note that then $g_i - z_i^p = -h_i^p z_i$. Define $\pi : X \to X_0$ to be the normalization of X_0 inside L, and set $H = \pi^* H_0$ (note, H is ample since π is finite).

Claim 12. *$\pi^*\alpha$ is contained in the subspace $H^1(X, \mathcal{O}_X(-H)) \subseteq H^1(X, \mathcal{O}_X)$ (note that $H^0(X, \mathcal{O}_X)$ surjects onto $H^0(H, \mathcal{O}_H)$).*

Proof. We set $V_i := \pi^{-1}(U_i)$. Now, $z_i \in \Gamma(V_i, \mathcal{O}_X)$ since z_i satisfies a monic equation with coefficients in $H^0(X_0, \mathcal{O}_{X_0})$. This implies that

$$\begin{aligned} \pi^*\alpha = & \quad [\alpha_{ij}] \\ = & \quad [\alpha_{ij} - z_i + z_j] \end{aligned}$$

so that

$$\begin{aligned} \left(\frac{\alpha_{ij} - z_i + z_j}{h_i}\right)^p &= \frac{\alpha_{ij}^p - z_i^p + z_j^p}{h_i^p} \\ &= \frac{(g_i - g_j) - z_i^p + z_j^p}{h_i^p} \\ &= \frac{(g_i - z_i^p) - (g_j - z_j^p)}{h_i^p} \\ &= -z_i + (h_j/h_i)^p z_j \\ &\in \Gamma(V_i \cap V_j, \mathcal{O}_X) \end{aligned}$$

But this implies that $\left[\frac{\alpha_{ij} - z_i + z_j}{h_i}\right] \in \Gamma(V_i \cap V_j, \mathcal{O}_X)$ which itself implies that $\alpha = [\alpha_{ij} - z_i - z_j] \in \Gamma(V_i \cap V_j, \mathcal{O}_X(H))$ and the claim follows. $\square$

The result then follows by the following lemma.

Lemma 28.3. [Mum67, Lemma 5] *Let $\pi : X' \to X$ be a finite surjective morphism of normal varieties over $k = \overline{k}$ such that $K(X) \subseteq K(X')$ is separable. Suppose that $\alpha \in H^1(X, \mathcal{O}_X)$ is such that $F(\alpha) = 0$ and $0 = \pi^*\alpha \in H^(X', \mathcal{O}_{X'})$. Then $\alpha = 0$.*

Proof. As before, represent α as $\{\alpha_{ij}\}$ in Čech cohomology for some cover U_i of X. Again we have $\alpha_{ij}^p = g_i - g_j$ with $g_i \in \Gamma(U_i, \mathcal{O}_{X_0})$. Because $\pi^*(\alpha) = 0$ there exists functions $h_i \in \Gamma(\pi^{-1}(U_i), \mathcal{O}_{X'})$ such that $\pi^*(\alpha_{ij}) = h_i - h_j$. Therefore,

$$h_i^p - \pi^*(g_i) = h_j^p - \pi^*(g_j).$$

Thus there exists a $\beta \in \Gamma(X', \mathcal{O}_{X'})$ such that $f^*(g_i) = h_i^p + \beta$ for all i. This implies that $\pi^*(g_i) \in K(X')^p$, which implies that $g_i \in K(X)^p$ for all i since $K(X) \subseteq K(X')$ is separable. Write $g_i = f_i^p$, $f_i \in K(X)$, and then since X is normal, we have that $f_i \in \Gamma(U_i, \mathcal{O}_X)$. Then, $a_{ij} = f_i - f_j$ since $a_{ij}^p = g_i - g_j$. This implies $\alpha = 0$ as desired. □

Remark 28.4. While there is no guarantee that X is smooth,

We now discuss Kawamata-Viehweg vanishing in positive characteristic.

Theorem 28.5. [Kaw82], [Vie82] *Suppose that X is a normal projective algebraic variety over an algebraically closed field of characteristic zero, B an effective $\mathbb{Q}$-divisor on X and D a Cartier (or $\mathbb{Q}$-Cartier integral) divisor. Assume that (X, B) is Kawamata log terminal and that $H = D - (K_X + B)$ is ample. Then $H^i(X, D) = 0$ holds for an $i > 0$.*

We will show that many varieties fail this, at least if they are constructed out of bizarre curves, we follow [Xie07].

Definition 28.6. [Tan72] Suppose that C is a smooth curve and $f \in K(C)$. Define

$$n(f) = \deg \lfloor \frac{1}{p} D(df) \rfloor.$$

Here $D(df)$ is the divisor associated to $df \in \omega_C$. The *Tango invariant* of C is defined to be

$$n(C) = \max\{n(f) | f \in K(C), f \notin (K(C))^p\}.$$

A curve C is called a *Tango curve* if $n(C) > 0$.

Before continuing, I'd like to discuss why Hiroshi Tango considered this notion, we will not include the proof at this time.

Theorem 28.7. [Tan72] *Let C be a curve of genus $g > 0$ with Tango invariant $n(C)$, then:*

(i) *For any line bundle $\mathscr{L}$ such that $\deg L > n(C)$, the Frobenius map $H^1(C, \mathscr{L}^{-1}) \to H^1(C, F^*\mathscr{L}^{-1})$ is injective (dually, $H^0(C, (F_*\omega_C) \otimes \mathscr{L}^p) \to H^0(C, \omega_C \otimes \mathscr{L})$ is surjective).*

(ii) *If $n(X) > 0$, then there exists a line bundle $\mathscr{M}$ of degree $n(C)$ such that the Frobenius map $H^1(X, \mathscr{M}^{-1}) \to H^1(X, F^*\mathscr{M}^{-1})$ is not injective.*

Remark 28.8. The Tango invariant of $\mathbb{P}^1$ is -1.

Example 28.9. [Tan72] The following curve $x^3y + y^3z + z^3x = 0$ in $\mathbb{P}^2$ is a genus 3 smooth Tango curve in characteristic 3. The partial derivatives are z^3, x^3, y^3 and so it is indeed smooth. Choose $f = (x - y)/z \in K(C)$. At the point $(0, 0, 1)$, we see that f vanishes to order 1, and so f is not in $K(C)^3$. One can show that

$$D(df) = -3(0,0,1) - 3(1,0,0) + \sum_{\alpha\alpha^3 = \alpha+1} \lambda(1-\alpha, -1, 1) + \text{ other positive terms.}$$

where $\lambda \geq 3$. $n(f) \geq 1$.

Assuming $f \notin (K(C))^p$, $df \neq 0$ so that $D(df) \sim K_C$ and has degree $2g-2$ where $g = g(C)$ is the genus of C. Also notice that $n(C) \leq \lfloor (2g-2)/p \rfloor$, thus $n(C) > 0$ implies that $g > 1$. There are many examples of Tango curves.

We have the following two short exactly sequences (just like we explored in the proof of Hara's lemma):

$$0 \to \mathcal{O}_C \to F_*\mathcal{O}_C \to \mathcal{B}^1 \to 0$$
$$0 \to \mathcal{B}^1 \to F_*\Omega_C \to \Omega_C \to 0$$

Here $\mathcal{B}^1$ is the image of $d : F_*\mathcal{O}_C \to F_*\Omega_C$.

Lemma 28.10. [Xie07] *With notation as above let L be a divisor on C, then $H^0(C, \mathcal{B}^1(-L)) = \{df | f \in K(C), D(df) \geq pL\}$. Furthermore, $n(C) > 0$ if and only if there exists an ample divisor L on C such that $H^0(C, \mathcal{B}^1(-L)) \neq 0$.*

Proof. Twisting the second equation above by $-L$ we get

$$0 \to \mathcal{B}^1(-L) \to F_*(\Omega_C(-pL)) \to \Omega_C(-L) \to 0.$$

Now, $H^0(C, \Omega_C(-pL)) = \{\omega \in \Omega_C | D(\omega) \geq pL\}$, so that

$$H^0(C, \mathcal{B}^1(-L)) = \{df | f \in K(C), D(df) \geq pL\}.$$

For the second statement, assume that $n(C) > 0$, thus there exists an $f_0 \in K(C)$ such that $n(f_0) = \deg \lfloor D(df_0)/p \rfloor > 0$. Let $L = \lfloor D(df_0)/p \rfloor$. Certainly $\deg L > 0$ and $D(df_0) \geq pL$ and so $df_0 \in H^0(C, \mathcal{B}^1(-L)) \neq 0$ as desired. The converse direction merely reverses this. $\square$

Using Tango curves, Raynaud constructed a smooth counterexample to Kodaira vanishing in each characteristic. These ideas have recently been further explored by Xie, and we have the following theorem.

Theorem 28.11. [Xie07] *Suppose that C is a tango curve, then there exists a $\mathbb{P}^1$-bundle $f : X \to C$ an effective $\mathbb{Q}$-divisor B and an integral divisor D on X such that (X, B) is KLT (in fact, B has SNC support with coefficients < 1) and $H = D - (K_X + B)$ is ample but $H^1(X, D) = 0$.*

Proof. This is taken from [Xie07]. We choose a divisor L on C such that $\deg L > 0$ and $H^0(C, \mathcal{B}^1(-L)) \neq 0$. Set $\mathscr{L} = \mathcal{O}_C(L)$, we then obtain

$$0 \to H^0(C, \mathcal{B}^1(-L)) \to H^1(C, \mathscr{L}^{-1}) \to H^1(C, \mathscr{L}^{-p}).$$

Choose $\alpha \in H^0(C, \mathcal{B}^1(-L))$ with image $\overline{\alpha} \in H^1(C, \mathscr{L}^{-1}) \cong \operatorname{Ext}^1_C(\mathscr{L}, \mathcal{O}_C)$. Thus we obtain an extension

$$0 \to \mathcal{O}_C \to \mathscr{E} \to \mathscr{L} \to 0.$$

Apply F^* and obtain

$$0 \to \mathcal{O}_C \to F^*\mathscr{E} \to \mathscr{L}^p \to 0$$

which corresponds to the extension class of $F^*\overline{\alpha}$, but this class is zero...

Let $f : X = \mathbb{P}(\mathscr{E}) \to C$ be the $\mathbb{P}^1$ bundle over C, with associated $\mathcal{O}_X(1)$ and fiber G. The surjection $\mathscr{E} \to \mathscr{L} \to 0$ induces a section $\sigma : C \to X$ by [Har77, IV, Prop 2.6] with image E. Furthermore, $f^*\mathcal{O}_C = \mathcal{O}_X \cong \mathcal{O}_X(1) \otimes \mathcal{O}_X(-E)$ so that $\mathcal{O}_X(E) = \mathcal{O}_X(1)$. We use the fact the sequence above is split and then and obtain:

$$0 \to \mathcal{O}_C \to (F^*\mathscr{E}) \otimes \mathscr{L}^{-p} \to \mathscr{L}^{-p} \to 0.$$

Thus we have the composition

$$H^0(C, \mathcal{O}_C) \to H^0(C, (F^*\mathscr{E}) \otimes \mathscr{L}^{-p}) \to H^0(C, S^p(\mathscr{E}) \otimes \mathscr{L}^{-p}) \cong H^0(X, \mathcal{O}_X(p) \otimes f^*\mathscr{L}^{-p}).$$

Thus we have a section $t \in H^0(X, \mathcal{O}_X(p) \otimes f^*\mathscr{L}^{-p})$ (corresponding to the image of 1). Therefore, we have a curve C' on X with $\mathcal{O}_X(C') \cong \mathcal{O}_X(p) \otimes f^*\mathscr{L}^{-p}$.

Claim 13. *We claim that C' is smooth and also that $C' \cap E = \emptyset$.*

Proof. We won't work out the details, but only sketch some evidence. Certainly $C'.E = (pE - p(\deg L)G).E = pE^2 - p(\deg L)$ where E^2 is the degree of $\mathscr{E}$ which is clearly $\deg L$. Thus as long as C' is irreducible, the second claim is obvious.

In fact, E and C' both correspond to splittings onto distinct terms of the split exact sequence

$$0 \to \mathcal{O}_C \to F^*\mathscr{E} \to \mathscr{L}^p \to 0.$$

compare with [Har77, Chapter V, Exercise 2.2]. □

Choose c a rational number satisfying $1/p < c < 1$ such that $cp \notin \mathbb{Z}$. Set $q = \lfloor cp \rfloor - 1$, and note that $q \geq 0$. Set $B = cC'$ and $D = qE + f^*(K_C - qL)$. Then

$$\begin{aligned} H &= D - (K_X + B) \\ &\equiv (\lfloor cp \rfloor - 1)E + f^*(K_C - qL) - K_X - cC' \\ &\equiv (\lfloor cp \rfloor - 1)E + f^*(K_C - (\lfloor cp \rfloor - 1)L) - (-2E + f^*K_C - f^*L) - c(pE - pf * L) \\ &\equiv (\lfloor cp \rfloor + 1 - cp)E + (cp - \lfloor cp \rfloor)f^*L. \end{aligned}$$

In particular, E is relatively ample and thus H is also ample. Clearly (X, B) is KLT.

Now, we need to show that $H^1(X, D) \neq 0$. Now, $D.G \geq 0$, thus by [Har77, Lemma 2.4], $R^1 f_*\mathcal{O}_X(D) = 0$ and $f_*\mathcal{O}_X(D)$ is locally free. Then

$$\begin{aligned} &H^1(X, D) \\ &= H^1(C, f_*\mathcal{O}_X(D)) \\ &= H^0(C, (f_*\mathcal{O}_X(D))^\vee \otimes \omega_C)^\vee \\ &= H^0(C, (f_*\mathcal{O}_X(D - f^*K_C))^\vee)^\vee \\ &= H^0(C, \mathcal{O}_C(qE - qL)^\vee)^\vee \\ &= H^0(C, (S^q(\mathscr{E})^\vee \otimes \mathscr{L}^q))^\vee. \end{aligned}$$

Now $\mathscr{L}^q$ is a quotient of $S^q(\mathscr{E})$, so $\mathscr{L}^{-q}$ is a subsheaf of $S^q(\mathscr{E})^\vee$. Thus,

$$H^1(X,D)^\vee = H^0(C, S^q(\mathscr{E})^\vee \otimes \mathscr{L}^q) \supseteq H^0(C, \mathscr{L}^{-q} \otimes \mathscr{L}^q) = H^0(C, \mathcal{O}_C) = k$$

proving the theorem. □

Q. Xie also proves the following result:

Theorem 28.12. [Xie07] *If there is a counter-example to the Kawamata-Viehweg vanishing theorem on a ruled surface $f : X \to C$, then either C is a Tango curve or all sections are ample.*

He also conjectures the following:

Conjecture 28.13. *If there is a counter-example to the Kawamata-Viehweg vanishing theorem on a normal projective surface X, then there exists a dominant rational map f from X to a smooth projective Tango curve C.*

29. Fujita's conjecture

We begin with a discussion of Castlenuovo-regularity, see [Laz04, Section 1.8].

Definition 29.1. Let $\mathscr{F}$ be a coherent sheaf on a projective variety X with a given ample line bundle $\mathscr{A} = \mathcal{O}_X(A)$ which is generated by global sections. A coherent sheaf $\mathscr{F}$ on X is called *m-regular* with respect to $\mathscr{A}$ if

$$H^i(X, \mathscr{F} \otimes \mathscr{A}^{\otimes(m-i)}) = 0$$

for $i > 0$.

Theorem 29.2 (Mumford)**.** [Laz04, Theorem 1.8.5] *With notation as above, suppose that $\mathscr{F}$ is an m-regular sheaf. Then $\mathscr{F} \otimes \mathscr{A}^m$ is globally generated.*

Example 29.3. [Laz04, Example 1.8.23]. Suppose that X is a smooth (or log canonical) n-dimensional variety of characteristic zero and $\mathscr{A} = \mathcal{O}_X(A)$ is an ample line bundle on X.

Now, for each $k \geq$ set $\mathscr{F}_k = \omega_X$. Clearly,

$$H^i(X, \mathscr{F}_k \otimes A^{\otimes(n+k-i)}) = 0$$

by Kodaira vanishing for any $k \geq 0$ and any $i > 0$. Thus ω_X is $n+k$-regular for all $k \geq 1$.

Applying the theorem above implies that $\mathcal{O}_X(K_X + (n+k)A)$ is globally generated for any $k \geq 1$.

Conjecture 29.4 (Fujita)**.** [Fuj87] *Suppose that $\mathscr{A}$ is an ample line bundle on a smooth n-dimensional variety X. Then:*

(i) $\omega_X \otimes \mathscr{A}^{n+1+k}$ *is globally generated for $k \geq 0$.*
(ii) $\omega_X \otimes \mathscr{A}^{n+2+k}$ *is very ample for $k \geq 0$.*

While we showed that (i) holds under the hypotheses that $\mathscr{A}$ is globally generated, condition (ii) also holds under the same condition, see [Laz04]. There a numerous refinements of this theorem by many authors including Angehrn, Demailly, Helmke, Kawamata, Kollár, Lazarsfeld, Seunghun Lee, Matsushita, Siu, Tsuji, and many others and has spawned much research in regards to Seshadri constants. It has been shown in characteristic zero in up through dimension 4, notably in [Rei88], [EL93], [Kaw97].

In characteristic $p > 0$, much less is known.

Theorem 29.5. [Smi97b], *cf* [Har05] *Suppose that X is a variety over $k = \bar{k}$. If X is only F-rational and $\mathscr{A}$ is globally generated then (i) holds in characteristic $p > 0$.*

The proof uses tight-closure methods, and we will prove it shortly.

Theorem 29.6. [Kee08] *Suppose that X is a variety over $k = \bar{k}$. If X is smooth and $\mathscr{A}$ is globally generated, then (ii) holds in characteristic $p > 0$.*

The proof uses Arapura's theory of Frobenius amplitude, which can be thought of as a means to measure positivity of line bundles and other sheaves in positive characteristic.

We now turn to the proof of (i) in positive characteristic, we follow Hara's approach from [Har05]. First recall the following definition.

Definition 29.7. Given an ideal $\mathfrak{a}$ in a ring R and an integer $t > 0$, the *test submodule* $\tau(\omega_R, \mathfrak{a}^t)$ is defined to be the unique smallest submodule $J \subseteq \omega_R$ such that

$$\Phi^e_R(F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil} J) \subseteq J$$

where $\Phi_R : F_*\omega_R \to R$ is the dual of Frobenius. It is also harmless to replace $\mathfrak{a}^{\lceil t(p^e-1)\rceil}$ by $\overline{\mathfrak{a}^{\lceil t(p^e-1)\rceil}}$ in the previous equation.

Given an appropriate test element $c \in R$, we still have

$$\tau(\omega, \mathfrak{a}^t) = \sum_{e \geq 0} \Phi^e_R(F^e_* c\overline{\mathfrak{a}^{\lceil tp^e\rceil}}\omega_R)$$

Any difference between using integral closures or not (or tp^e vs $t(p^e-1)$ can be absorbed into the c-term.

Lemma 29.8. [Har05, Proposition 2.4], [HT04] *Assume that R is a $\mathbb{N}$-graded ring of dimension $d \geq 1$ and further suppose that R has a graded system of parameters in degree 1. Set $\mathfrak{m} = R_+$. Then if $l \geq 0$ is an integer, we have*

$$\tau(\omega_R, \mathfrak{m}^{l+d-1}) = \overline{\mathfrak{m}^l}\tau(\omega_R, \mathfrak{m}^{d-1}).$$

Proof. The proof is essentially the same as a proof in [BSTZ10]. Choose $\mathfrak{a}$ to be the ideal generated by our given system of parameters noting that $\overline{\mathfrak{a}} = \mathfrak{m}$ (in particular, it is generated by d-elements). We consider the dual of Frobenius, $\Phi_R : F_*\omega_R \to \omega_R$. We then note the following equality,

$$\mathfrak{m}^{p^n(l+d-1)} = \mathfrak{a}^{p^n(l+d-1)} = \mathfrak{a}^{p^n l}\mathfrak{a}^{p^n(d-1)} = (\mathfrak{a}^{[p^n]})^l \mathfrak{a}^{p^n(d-1)}.$$

Then for some $w > 0$ and appropriate $0 \neq c \in R$ we have:

$$\tau(\omega_R, \mathfrak{a}^{l+d-1}) = \sum_{n=1}^{w} \Phi_R^n(F_*^n \mathfrak{a}^{(l+d-1)p^n} c\omega_R), \text{ and}$$

$$\tau(\omega_R, \mathfrak{a}^{l+d-1}) = \sum_{n=1}^{w} \Phi_R^n(F_*^n \overline{\mathfrak{a}^{(l+d-1)p^n}} c\omega_R), \text{ and}$$

$$\tau(\omega_R; \mathfrak{a}^{d-1}) = \sum_{n=1}^{w} \Phi_R^n(F_*^n \mathfrak{a}^{(d-1)p^{ne}} c\omega_R).$$

However,

$$\begin{aligned}
\tau(\omega_R, \mathfrak{a}^{l+d-1}) &= \sum_{n=1}^{w} \Phi_R^n(F_*^n \mathfrak{a}^{(l+d-1)p^n} c\omega_R) \\
&= \sum_{n=1}^{w} \Phi_R^n(F_*^n (\mathfrak{a}^{[p^n]})^l \mathfrak{a}^{p^n(d-1)} c\omega_R) \\
&= \sum_{n=1}^{w} \Phi_R^n(F_*^n \overline{(\mathfrak{a}^l)}^{[p^n]} \mathfrak{a}^{p^n(d-1)} c\omega_R) \\
&= \overline{(\mathfrak{a}^l)} \sum_{n=1}^{w} \phi_n(F_*^n \mathfrak{a}^{p^n(d-1)} c\omega_R) \\
&= \overline{\mathfrak{m}^l} \tau(\omega_R; \mathfrak{a}^{d-1})
\end{aligned}$$

as desired. □

Lemma 29.9. [Har05, Lemma 2.6] *Suppose that R is a d-dimensional normal graded ring over a perfect field $k = R_0$ of characteristic $p > 0$ with $\mathfrak{m} = R_+$ and also that R has a system of parameters of degree 1. Suppose further that R is F-rational on the punctured spectrum. Then*

$$\tau(\omega, \mathfrak{m}^l) = [\omega_R]_{>l}$$

for $l \gg 0$.

Proof. We will work in the Matlis dual world. The Matlis dual of $\omega_R/\tau(\omega, \mathfrak{m}^l)$ is $0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)}$ and so we want to show that

$$0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)} = H^d_\mathfrak{m}(R)_{\geq -l}$$

Recall that $0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)}$ is the set of elements $z \in H^d_\mathfrak{m}(R)$ such that there exists $0 \neq c \in R$ satisfying $c\overline{\mathfrak{m}^{lp^e}} z^{p^e} = 0 \in H^d_\mathfrak{m}(R)$.

So we have two containments to show. First suppose that $z \in H^d_\mathfrak{m}(R)_{\geq -l}$. Thus $\mathfrak{m}^{lp^e} z^{p^e} \in H^d_\mathfrak{m}(R)_{\geq 0}$, but $H^d_\mathfrak{m}(R)_{\geq 0}$ has finite length and so there is a non-zero element of R which annihilates it, which implies $z \in 0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)}$.

The reverse containment is somewhat more involved. First note that because $\omega_R/\tau(\omega, \mathfrak{m}^l)$ has support at the maximal ideal, $0^{*\mathfrak{m}^l}_{H^d_{\mathfrak{m}}(R)}$ has finite length. This implies that the Frobenius map

$$F^e : [H^d_{\mathfrak{m}}(R)]_{<-l} \to [H^d_{\mathfrak{m}}(R)]_{<-p^e l}$$

is injective for $l \gg 0$.

Choose $0 \neq z \in [H^d_{\mathfrak{m}}(R)]_{<-l}$. Therefore, $\lim_{e\to\infty} \deg(z^{p^e}) + lp^e = -\infty$.

Claim 14. *For $e \gg 0$, there exists a sequence of $c_e \in R$ such that $\lim_{e\to\infty} \deg(c_e) = \infty$ and such that $c_e R_{p^e l} z^{p^e} \neq 0$.*

Proof. The socle of $H^d_{\mathfrak{m}}(R)$ is the set of elements of $H^d_{\mathfrak{m}}(R)$ annihilated by $\mathfrak{m}$. This is a module of finite length since its Matlis dual is $\omega_R/(\mathfrak{m}\omega_R)$. To see this, given a set of generators y_i of $\mathfrak{m}$, the socle is the kernel of $H^d_{\mathfrak{m}}(R) \to \oplus y_i H^d_{\mathfrak{m}}(R)$. Matlis duality gives the claim. Likewise the module of elements of $H^d_{\mathfrak{m}}(R)$ annihilated by $R_{\geq n}$ is also finite length for any n.

Now, $R_{p^e l} z^{p^e}$ is non-zero for $e \gg 0$ because if it was zero, then $R_{p^e l-1} z^{p^e}$ would be in the socle or zero. Inductively, this is ridiculous. Thus we can find c_e satisfying the desired properties. $\square$

Using the fact that the degrees of c_e are increasing, it then follows (by arguments I won't repeat here, see the citation for more descriptions, or [Sch08a]) that c_e is a "test element" for $e \gg 0$. We also know that $e \gg 0$, $c_e \overline{\mathfrak{m}}^{p^e l} z^{p^e} = c_e R_{p^e l} z^{p^e} \neq 0$, which implies that $z \notin 0^{*\mathfrak{m}^l}_{H^d_{\mathfrak{m}}(R)}$. This completes the proof. $\square$

We need one more lemma.

Lemma 29.10 (Smith)**.** *With notation as above, $\omega_X \otimes \mathscr{L}^{\otimes m}$ is globally generated if $[\omega_R]_l = R_{l-m}[\omega_R]_m$ for all $l \gg 0$.*

Proof. Suppose first the condition is satisfied, but that $\omega_X \otimes \mathscr{L}^{\otimes m}$ is not globally generated. In particular, the global sections of $\omega_X \otimes \mathscr{L}^{\otimes m}$ all vanish on some closed subvariety. But then $R_{l-m}[\omega_R]_m$ vanishes on that same subvariety for $l \gg 0$. $\square$

Now we turn to our main result of this section: Suppose that $\mathscr{A}$ is an ample and globally generated line bundle on a smooth n-dimensional variety X. Then $\omega_X \otimes \mathscr{A}^{n+1+k}$ is globally generated for $k \geq 0$.

Proof of Theorem 29.5. This is taken from [Har05, Theorem 2.1]. Set $R = R(X, \mathscr{A})$ and set $d = n+1 = \dim X + 1 = \dim R$. As before, set $\mathfrak{m} = R_+$ and observe that $\overline{\mathfrak{m}^l} = R_{\geq l}$.

Now, we have the following inclusions for $l \gg 0$:

$$R_{l-d}[\omega_R]_{d-1} \subseteq [\omega_R]_{>l-1} = \tau(\omega, \mathfrak{m}^{l-1}) = R_{\geq l-d}\tau(\omega, \mathfrak{m}^{d-1}) \subseteq R_{\geq l-d}[\omega_R]_{>d-1}$$

This completes the proof. $\square$

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