# The `TestIdeals` package for Macaulay2

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# History

- Consider rings $R$ of characteristic $p > 0$.
- No resolution of singularities (in general).
- Kunz proved:

**Theorem (Kunz)**

*$R$ is regular if and only if Frobenius is flat.*

- How flat is Frobenius?

**Definition (Hochster-Roberts, Mehta-Ramanathan)**

$R$ is *F-pure* if and only if $R \to R^{1/p^e}$ splits.

- $F$-pure is analogous to (semi)log canonical singularities. [Hara-Watanabe]

# Fedder

Checking $F$-purity can be pretty easy.

- Fedder's Criterion. $R = S/I$, $S$ is polynomial.

**Theorem (Fedder)**

*$R$ is $F$-pure at $\mathfrak{m}$ if and only if $I^{[p]} : I \not\subseteq \mathfrak{m}^{[p]}$.*

- If $I = (f)$, then $I^{[p]} : I = (f^{p-1})$. (*BOARD*)
- For example.

```
i5 : S = ZZ/7[x,y,z];
i6 : f = x^3 + y^3 + z^3;
i8 : isSubset(ideal(f^6), ideal(x^7, y^7, z^7))
o8 = false
```

We have written a package `TestIdeals.m2` that computes whether a ring (or pair) is:

- *F*-pure
  - Analog of SLC.
- *F*-regular
  - Analog of KLT.
- *F*-rational
  - Analog of rational.
- *F*-injective
  - Analog of Du Bois.
- Test ideals
  - Analogs of multiplier ideals
- *F*-pure thresholds (with `FThresholds.m2`).
  - Analogs of log canonical thresholds.

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# Fedder, part 2

- Fedder's criterion works because maps

$$\phi_R : R^{1/p^e} \to R$$

come from maps

$$\phi_S : S^{1/p^e} \to S$$

such that $\phi_S(I^{1/p^e}) \subseteq I$.

- In fact,

$$I^{[p^e]} : I \cong \{\phi \in \mathrm{Hom}_S(S^{1/p^e}, S) \mid \phi(I^{1/p^e}) \subseteq I\}.$$

- Translates questions on $R$ to polynomial ring $S$.

# Frobenius trace

One more big tool.

- There exists $\Phi : S^{1/p^e} \to S$.
- $\Phi\left(x_1^{\frac{p^e-1}{p^e}} \cdots x_n^{\frac{p^e-1}{p^e}}\right) = 1$
- Other monomials to 0.
- $\Phi$ generates $\mathrm{Hom}_S(S^{1/p^e}, S)$.
- $\Phi$ is Grothendieck dual to Frobenius.
- $\Phi(J^{1/p^e}) \subseteq I$ if and only if

$$I^{[p^e]} \subseteq J.$$

**Theorem (Fedder restated)**

$$\Phi((I^{[p^e]} : I)^{1/p^e}) \equiv_I \mathrm{Image}(\mathrm{Hom}_R(R^{1/p^e}, R) \xrightarrow{@1} R)$$

*defines locus where $R = S/I$ is not $F$-pure.*

# Implementation

We compute some Macaulay2 examples. $\Phi(J)$ is called the *Frobenius root of J*.

```
i12 : I =ideal(x^3 + y^3 + z^3);
i13 : frobeniusRoot(1, I^7 : I)
o13 = ideal 1
i14 : isFPure(S/I)
o14 = true
i15 : J = ideal(x^4+y^4+z^4);
i16 : frobeniusRoot(1, J^7 : J)
              2             2       2
o16 = ideal (z , y*z, x*z, y , x*y, x )
i19 : isFPure(S/J)
o19 = false
```

# More examples

```
i20 : T = ZZ/5[a,b,c,d,e];
i21 : B = ZZ/5[x,y];
i22 : f = map(B, T, {x^4, x^3*y, x^2*y^2, x*y^3, y^
                     4    3     2  2      3    4
o22 = map(B,T,{x , x y, x y , x*y , y })
o22 : RingMap B <--- T
i23 : I = ker f
                 2                                2
o23 = ideal (d  - c*e, c*d - b*e, b*d - a*e, c  - a
o23 : Ideal of T
i24 : isFPure(T/I)
o24 = true
```

- Analog of KLT.

**Definition**

$R$ is *strongly $F$-regular* if for every (interesting[1]) $c \in R$, there is some $e$ and $\phi : R^{1/p^e} \to R$ so that $\phi(c^{1/p^e}) = 1$.

- If translated by Fedder's methods,

**Theorem**

*$R = S/I$ is strongly $F$-regular if and only if*

$$I + \Phi((c(I^{[p^e]} : I))^{1/p^e}) = S.$$

# $F$-regularity checking

```
i3 : S = ZZ/7[x,y,z];
i4 : R = S/ideal(x^2-y*z)
i5 : isFRegular(R);
o5 = true
i20 : A = ZZ/7[x,y,z]/(y^2*z - x*(x-z)*(x+z));
i21 : C = ZZ/7[a,b,c,d,e,f];
i22 : g = map(A, C, {x^2, x*y, x*z, y^2, y*z, z^2})
i23 : I = ker g;
i26 : isFRegular(C/I);
o26 = false
```

- We can only show that $\mathbb{Q}$-Gorenstein rings are not $F$-regular.
- The `QGorensteinIndex=>infinity` option can prove a non-$\mathbb{Q}$-Gorenstein ring is $F$-regular.

# $F$-regularity of pairs

```
i3 : S = ZZ/7[x,y,z];
i4 : R = S/ideal(x^2-y*z)
i6 : h = y;
i7 : isFRegular(1/2, y)
o7 = false
i8 : isFRegular(1/3, y)
o8 = true
```

- The pair $(R, h^{1/2})$ is not $F$-regular but $(R, h^{1/3})$ is.
- The `FThresholds` package can even compute $F$-pure thresholds.

# $F$-rationality

- Analog of rational singularities.
- Implies (pseudo-)rational singularities in a fixed characteristic.
  - $\mathcal{O}_X \simeq R\pi_* \mathcal{O}_Y$
- Here's our definition:

**Definition**

$R$ has *$F$-rational singularities* if it is

- Cohen-Macaulay
- $(c^{1/p^e} \cdot \omega_{R^{1/p^e}}) \xrightarrow{F^e-\text{dual}} \omega_R$ surjects.

# $F$-rational examples

Here is an example of an $F$-rational (but not $F$-regular) ring.

```
i8 : S = ZZ/3[a,b,c,d,t]; m = 4; n = 3;
i11 :  M = matrix{ {a^2 + t^m, b, d},
                   {c, a^2, b^n-d} };
              2       3
o11 : Matrix S  <--- S
i12 :  I = minors(2, M);
i13 : R = S/I;
i14 : isFRational(R)
o14 = true
```

Appeared in work of Anurag Singh (deform $F$-regularity)

# Characteristic zero applications

Characteristic $p > 0$ conclusions imply results in characteristic zero.

**Theorem (Ma-•)**

*Suppose $R$ is a ring of mixed characteristic finite type over $\mathbb{Z}$. Suppose $p \in \mathbb{Z}$ is a regular element and $Q \subseteq R$ is a prime not containing any nonzero prime of $\mathbb{Z}$ so that $(p) + Q \neq R$.*

*If $R/pR$ is $F$-rational, then $R_Q = R_Q \otimes \mathbb{Q}$ has rational singularities.*

- Analogous statement for log terminal/$F$-regular singularities, if the $\mathbb{Q}$-Gorenstein not divisible by $p$.
- Not known for log canonical/$F$-pure singularities (need mixed char inversion of adjunction).

# Test ideals

We can compute test ideals too. Including of pairs.

- In a $\mathbb{Q}$-Gorenstein ring.
- $\tau(R, f^t)$ equals sum of images of maps

$$\phi : (cf^{\lceil t(p^e-1)\rceil} R)^{1/p^e} \to R.$$

  $c$ as before. [Hara-Takagi]
- We use it to check $F$-regularity.
  - $(R, f^t)$ is $F$-regular if and only if $\tau(R, f^t) = R$.
- Trick is stabilize image sums above.
- Can compute parameter test modules and parameter test ideals too.

# Example

```
i2 : R = ZZ/5[x,y];
i3 : f = y^2-x^3;
          3    2
o3 = - x  + y
i4 : testIdeal(4/5, f);
o4 = ideal (y, x)
i5 : testIdeal(4/5-1/10000, f)
o5 = ideal 1
```

- We can compute $\tau(R, f^{t-\epsilon})$, which is used to compute jumping numbers and $F$-pure thresholds.
- Needs `HSLGModule` function.

# Thanks!

You can go to:

`http://www.math.utah.edu/~schwede/M2.html`

to try it yourself!

It's also built into the latest version of Macaulay2.