EXAMPLES OF ODD BEHAVIOR WITH SEMINORMALITY

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The purpose of this short exposition is to give several examples of odd behavior of seminormal schemes. All rings and schemes considered here will be assumed to be noetherian, although certain generalizations without that hypothesis apply. (Of course all rings are commutative with unity and maps of rings send 1 to 1). I am making no claim as the originality of these examples, see the references at the end of this document for a more complete picture.

First let us recall the definition of a seminormal scheme.

Definition 0.1. A finite birational map of reduced schemes $f : X \to Y$ is called *subintegral* if it is a bijection on points, and for every point $x \in X$ with $y = f(x)$, the induced map of residue fields $k(y) \to k(x)$ is an isomorphism. We call a map of rings subintegral if it induces a subintegral map of schemes.

Remark 0.2. Usually subintegral refers only to extensions of rings $R \subset S$, and Greco and Traverso [GT80] used the word *quasi-isomorphism* for such a map of schemes. I use the word subintegral here since it is commonly used in the ring-theoretic setting.

There is one other characterization of subintegrality which first appeared in [Ham75], but also see [Swa80] and [LV81a].

Theorem 0.3. *Let $A \subset B$ be a ring extensions. Then A is subintegral in B if every element $b \in B$ such that $b^2, b^3 \in A$ also satisfies $b \in A$.*

Definition 0.4. A reduced scheme X is called *seminormal* if every subintegral map $Y \to X$ is an isomorphism.

Note that a normal scheme is seminormal. Geometrically speaking, a scheme is seminormal if all the non-normality it has is due to gluing points (subschemes) together and that gluing is done as transversally as possible.

A common misconception is that one needs only consider bijective maps. This is not the case if one does not work over an algebraically closed field of characteristic zero, as the following examples show. One also often needs the finiteness hypothesis.

Example 0.5. *Two different seminormal schemes over a field of characteristic zero with a finite bijective birational map between them*

There are several ways to construct seminormal varieties with a finite bijective map between them. Consider the rings with inclusion map $\mathbb{R}[x, ix] \subset \mathbb{C}[x]$. The two rings are clearly birational and the map is clearly finite. They are also both finite type over a field of characteristic zero. I claim that map between them is bijective. Consider the $\operatorname{Spec} \mathbb{R}[x, ix] - (x, ix)$ (the punctured plane). It is easy to see that this is just $\operatorname{Spec} \mathbb{C}[x]$ with the origin removed and map above induces the isomorphism. On the other hand, the ideal $(x) \subset \mathbb{C}[x]$ has inverse image (x, ix) in $\mathbb{R}[x, ix]$ and so the two maps are in fact bijective. It is not difficult to check that both schemes are seminormal. Note that the map above is not subintegral since the residue fields at the origin are $\mathbb{R}$ and $\mathbb{C}$ respectively.

It is not even enough to biject on points and have isomorphic residue fields on the closed points over an algebraically closed field.

Example 0.6. *Two different seminormal schemes over an algebraically closed field of positive characteristic with a finite birational bijective map between them*

Let k be an algebraically closed field of characteristic 2 (similar examples can be constructed for any field of positive characteristic). Consider the map of rings $k[x^2, xy, y] \subset k[x, y]$. The first ring is the coordinate ring of the pinch point. It clearly induces a birational finite map of schemes. Outside of the ideal $(y, xy) \subset k[x^2, xy, y]$, corresponding to $(y) \subset k[x, y]$ the map is clearly an isomorphism. However, if we mod out by those ideals we have $k[x^2] \subset k[x]$ which clearly bijects on points. In fact, we see immediately that the map induces an isomorphism of residue fields of the closed points (they are all k). Both schemes are easily seen to be seminormal and the map induced by the inclusion is not subintegral since the residue field at the generic point of $(y, xy) \subset k[x^2, xy, y]$ is $k(x^2)$ and the residue field at the corresponding point $(y) \subset k[x, y]$ is $k(x)$.

The finiteness requirement in the definition in subintegral is necessary as well, as the following example of [Vit87] indicates.

Example 0.7. *Two different birational seminormal varieties over an algebraically field of characteristic zero, with a bijective map between them*

Let C be a nodal curve and $\overline{C}$ be it's normalization. Let $x, y \in \overline{C}$ be the two points that lie over the node. Then the map $\overline{C} \backslash \{x\} \to C$ is a bijective birational map which induces isomorphisms of all residue fields between two *different* seminormal schemes. But this is ok, the map is not subintegral because it is not finite.

Remark 0.8. This sort of example only occurs in dimension 1, see [Vit87] for details.

There are examples of seminormal rings R with minimal primes p such that R/p is *not* seminormal. Probably the easiest example of this is found in [GT80].

Example 0.9. Let $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ and $Y = \mathbb{A}^2 = \operatorname{Spec} k[u, v]$. The idea is to glue X to Y along a closed subset. We choose the subset $L = \operatorname{Spec} k[x, y]/(y)$ of X and the subset $C = \operatorname{Spec} k[u, v]/(u^3 - v^2)$ of Y and we glue by the map

$$k[u, v]/(u^3 - v^2) \to k[x, y]/(y)$$

that sends u to x^2 and v to x^3. The coordinate ring of this gluing is

$$\frac{k[a, b, c, d, e]}{(b, a, c^3 - d^2 - e) \cap (e, bc - ad, c^3 - d^2, ac^2 - bd, a^2c - b^2, a^3d - b^3)}.$$

Let $I = (b, a, c^3 - d^2 - e)$ and $J = (e, bc - ad, c^3 - d^2, ac^2 - bd, a^2c - b^2, a^3d - b^3)$. It is easy to see that both I and J are prime (with a computer algebra program).

Note that $k[a, b, c, d, e]/I$ can be identified with $k[c, d]$ so it is just a copy of $\mathbb{A}^2$. On the other hand $k[a, b, c, d, e]/J$ is

$$k[a, b, c, d]/(bc - ad, c^3 - d^2, ac^2 - bd, a^2c - b^2, a^3d - b^3)$$

This can be thought of as $k[x, y]$ with the tangent space at the origin is killed in one direction (it can also be written as $k[y, xy, x^2, x^3]$). By basic facts about gluing it is easy to see that

$$R = k[a, b, c, d, e]/(I \cap J)$$

is seminormal (see [GT80] for another explanation), but it clearly has a minimal prime J, such that R/J is not seminormal.

In some sense the key point here was that when gluing X to Y, X had to be made non-seminormal because you were turning a line on X into a cuspidal curve.

References

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