First let us fix a small universe to work in. Let $\underline{\mathrm{Sch}}$ denote the category of reduced schemes. One should note that the usual fibred product of schemes $X \times_S Y$ need not be reduced, even when X and Y are reduced. We wish to construct the fibred product in the category of reduced schemes. Given any scheme W (reduced or not) with maps to X and Y over S, there is always a unique morphism $W \to X \times_S Y$. But, we have a natural unique morphism $W_{\mathrm{red}} \to (X \times_S Y)_{\mathrm{red}}$. Thus $(X \times_S Y)_{\mathrm{red}}$ is the fibred product in the category of reduced schemes.

Let us denote by $\underline{1}$ the category $\{0\}$ and by $\underline{2}$ the category $\{0 \to 1\}$. Let n be an integer ≥ -1. We denote by $\square_n^+$ the product of $n+1$ copies of the category $\underline{2} = \{0 \to 1\}$ [GNPP88, I, 1.15]. The objects of $\square_n^+$ are identified with the sequences $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \in \{0, 1\}$ for $0 \leq i \leq n$. For $n = -1$, we set $\square_{-1}^+ = \{0\}$ and for $n = 0$ we have $\square_0^+ = \{0 \to 1\}$. We denote by $\square_n$ the full subcategory consisting of all objects of $\square_n^+$ except the initial object $(0, \ldots, 0)$. Clearly, the category $\square_n^+$ can be identified with the category of $\square_n$ with an augmentation map to $\{0\}$.

Definition 0.1. A *diagram of schemes* is a functor F from a category $\mathscr{C}^{\mathrm{op}}$ to the category of schemes. A *finite diagram of schemes* is a diagram of schemes such that the aforementioned category $\mathscr{C}$ has finitely many objects and morphisms; in this case such a functor will be called a *$\mathscr{C}$-scheme*. A morphism of diagrams of schemes $F : \mathscr{C}^{\mathrm{op}} \to \underline{\mathrm{Sch}}$ to $G : \mathscr{D}^{\mathrm{op}} \to \underline{\mathrm{Sch}}$ is the combined data of a functor $\phi : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$ together with a natural transformation of functors f from F to $G \circ \phi$.

Remark 0.2. With the above definitions, the class of (finite) diagrams of schemes can be made into a category. Likewise the set of $\mathscr{C}$-schemes can also be made into a category (where the functor $\phi : \mathscr{C} \to \mathscr{C}$ is always the identity functor).

Remark 0.3. Let I be a category. If instead of a functor to the category of reduced schemes, one considers a functor to the category of topological spaces, or the category of categories, one can define I-topological spaces, and I-categories in the obvious way.

If $X_{\bullet} : I^{\mathrm{op}} \to \underline{\mathrm{Sch}}$ is an I-scheme, and $i \in I$, we denote by X_i the scheme corresponding to i. Likewise if $a \in I$ is a morphism $a : j \to i$, then X_a will denote the corresponding morphism $X_a : X_i \to X_j$. If $f : Y_{\bullet} \to X_{\bullet}$ is a map of I-schemes, we denote by f_i the induced map $Y_i \to X_i$. If $X_{\bullet}$ is an I-scheme, a closed sub-I-scheme is a morphism of I-schemes $g : Z_{\bullet} \to X_{\bullet}$ such that for each $i \in I$, the map $g_i : Z_i \to X_i$ is a closed immersion. We will often suppress the g of the notation as no confusion is likely to arise. More generally, any property of a morphism of schemes (projective, proper, separated, closed immersion, etc...) can be generalized to the notion of a morphism of I-schemes by requiring that for each object i of I, g_i has the desired property (projective, proper, separated, closed immersion, etc...)

Definition 0.4. [GNPP88, I, 2.2] Given a morphism of I-schemes $f : Y_{\bullet} \to X_{\bullet}$, we define the *discriminant of* f to be the smallest closed sub-I-scheme $Z_{\bullet}$ of $X_{\bullet}$ such that $f_i : (Y_i - (f_i^{-1}(Z_i))) \to (X_i - Z_i)$ is an isomorphism for all i.

Definition 0.5. [GNPP88, I, 2.5] Let $S_{\bullet}$ be an I-scheme, $f : X_{\bullet} \to S_{\bullet}$ a proper morphism of I-schemes, and $D_{\bullet}$ the discriminant of f. We say that f is a *resolution of* $S_{\bullet}$ if $X_{\bullet}$ is a smooth I-scheme (meaning that each X_i is smooth) and $\dim f_i^{-1}(D_i) < \dim S_i$, for all $i \in \operatorname{Ob} I$.

Remark 0.6. This is the definition found in [GNPP88], and essentially the one found in the book by Peters and Steenbrink, [PS08]. This is different from the one I gave in the seminar.

In particular, they have the additional requirement that $\dim f_i^{-1}(D_i) < \dim S_i$. They don't require that the maps are surjective (of course, the ones they construct in practice surjective).

Consider the following example: the map $k[x,y]/(xy) \to k[x]$ which sends y to 0. I claim that the associated map of schemes is a "resolution" of the $*$-scheme, $\operatorname{Spec} k[x,y]/(xy)$. The discriminant is $\operatorname{Spec} k[x,y]/(x)$. The pre-image however is simply the origin on $k[x]$, which has lower dimension than "1". Resolutions like this one are sometimes convenient to consider.

On the other hand, this definition seems to allow something it shouldn't. Choose any variety X of dimension greater than zero and a closed point $z \in X$. Consider the map $z \to X$ and consider the $*$-scheme X. The discriminant is all of X. However, the pre-image of X is still just a point, which has lower dimension than X itself, by hypothesis.

In view of these remarks, sometimes it is convenient to assume also that $\dim D_i < \dim S_i$ for each $i \in \operatorname{Ob} I$. In the resolutions of I-schemes that we construct (in particular, in the ones that are used to that prove cubic hyperresolutions exist), this always happens.

Let I be a category. The set of objects of I can be given the following pre-order relation, $i \leq j$ if and only if $\operatorname{Hom}_I(i,j)$ is nonempty. We will say that a category I is ordered if this pre-order is a partial order and, for each $i \in \operatorname{Ob} I$, the only endomorphism of i is the identity [GNPP88, I, C, 1.9]. Note that a category I is ordered if and only if all isomorphisms and endomorphisms of I are the identity.

It turns out of that resolutions of I-schemes always exist under reasonable hypotheses.

Theorem 0.7. [GNPP88, I, Theorem 2.6] *Let S be an I-scheme of finite type over a field k. Suppose that k is a field of characteristic zero and that I is a finite ordered category. Then there exists a resolution of S.*

In order to construct a resolution $Y_{\bullet}$ of an I-scheme $X_{\bullet}$, it might be tempting to simply resolve each X_i, set Y_i equal to that resolution, and somehow combine this data together. Unfortunately this cannot work, as shown by the example below.

Example 0.8. Consider the pinch point singularity,

$$X = \operatorname{Spec} k[x,y,z]/(x^2y - z^2) = \operatorname{Spec} k[s, t^2, st]$$

and let Z be the closed subscheme defined by the ideal (s, st) (this is the singular set). Let I be the category $\{0 \to 1\}$. Consider the I-scheme defined by $X_0 = X$ and $X_1 = Z$ (with the closed immersion as the map). X_1 is already smooth, and if one resolves X_0, (that is, normalizes it) there is no compatible way to map X_1 (or even another birational model of X_1) to it, since its pre-image by normalization will two-to-one onto $Z \subset X$! The way this problem is resolved is by creating additional components. So to construct a resolution $Y_{\bullet}$ we set $Y_1 = Z = X_1$ (since it was already smooth) and set $Y_0 = \overline{X}_0 \coprod Z$ where $\overline{X}_0$ is the normalization of X_0. The map $Y_1 \to Y_0$ just sends Y_1 (isomorphically) to the new component and the map $Y_0 \to X_0$ is the disjoint union of the normalization and inclusion maps.

One should note that although the theorem proving the existence of resolutions of I-schemes is constructive, [GNPP88], it is often easier in practice to construct an ad-hoc resolution.

Now that we have resolutions of I-schemes, we can discuss cubic hyperresolutions of schemes, in fact, even diagrams of schemes have cubic hyperresolutions! First we will discuss a single iterative step in the process of constructing cubic hyperresolutions. This step is called a 2-resolution.

Definition 0.9. [GNPP88, I, 2.7] Let S be an I-scheme and $Z_\bullet$ an $\square_1^+ \times I$-scheme. We say that $Z_\bullet$ is a 2-*resolution* of S if $Z_\bullet$ is defined by the cartesian square (pullback, or fibred product in the category of (reduced) I-schemes) of morphisms of I-schemes below

$$\begin{tikzcd} Z_{11} \arrow[r, hook] \arrow[d] & Z_{01} \arrow[d, "f"] \\ Z_{10} \arrow[r, hook] & Z_{00} \end{tikzcd}$$

where

i) $Z_{00} = S$.
ii) Z_{01} is a smooth I-scheme.
iii) The horizontal arrows are closed immersions of I-schemes.
iv) f is a proper I-morphism
v) Z_{10} contains the discriminant of f; in other words, f induces an isomorphism of $(Z_{01})_i - (Z_{11})_i$ over $(Z_{00})_i - (Z_{10})_i$, for all $i \in \operatorname{Ob} I$.

Clearly 2-resolutions always exist under the same hypotheses that resolutions of I-schemes exist: set Z_{01} to be a resolution, Z_{10} to be discriminant (or any appropriate proper closed sub-I-scheme that contains it), and Z_{11} its (reduced) pre-image in Z_{01}.

Consider the following example,

Example 0.10. Let $I = \{0\}$ and let S be the I-scheme $\operatorname{Spec} k[t^2, t^3]$. Let $Z_{01} = \mathbb{A}^1 = \operatorname{Spec} k[t]$ and $Z_{01} \to S = Z_{00}$ be the map defined by $k[t^2, t^3] \to k[t]$. The discriminant of that map is the closed subscheme of $S = Z_{00}$ defined by the map $\phi : k[t^2, t^3] \to k$ which sends t^2 and t^3 to zero. Finally we need to define Z_{11}. The usual fibered product in the category of schemes is $k[t]/(t^2)$, but we work in the category of reduced schemes, so instead the fibered product is simply the associated reduced scheme (in this case $\operatorname{Spec} k[t]/(t)$). Thus our 2-resolution is defined by the diagram of rings pictured below.

$$\begin{tikzcd} & k[t]/(t) & \\ k[t]/(t) \arrow[ur] & & k[t] \arrow[ul] \\ & k[t^2, t^3] \arrow[ul] \arrow[ur] & \end{tikzcd}$$

.

We need one more definition before defining a cubic hyperresolution,

Definition 0.11. [GNPP88, I, 2.11] Let r be an integer greater than or equal to 1, and let $X_\bullet^n$ be an $\square_n^+ \times I$-scheme, for $1 \le n \le r$. Suppose that for all n, $1 \le n \le r$, the $\square_{n-1}^+ \times I$-schemes $X_{00\bullet}^{n+1}$ and $X_{1\bullet}^n$ are equal. Then we define, by induction on r, a $\square_r^+ \times I$-scheme

$$Z_\bullet = \operatorname{rd}(X_\bullet^1, X_\bullet^2, \ldots, X_\bullet^r)$$

that we call the *reduction* of $(X^1_\bullet, \dots, X^r_\bullet)$, in the following way: If $r = 1$, one defines $Z_\bullet = X^1_\bullet$, if $r = 2$ one defines $Z_{\bullet\bullet} = \mathrm{rd}(X^1_\bullet, X^2_\bullet)$ by

$$Z_{\alpha\beta} = \begin{cases} X^1_{0\beta} & , \text{ if } \alpha = (0,0), \\ X^2_{\alpha\beta} & , \text{ if } \alpha \in \square_1, \end{cases}$$

for all $\beta \in \square_0^+$, with the obvious morphisms. If $r > 2$, one defines $Z_\bullet$ recursively as $\mathrm{rd}(\mathrm{rd}(X^1_\bullet, \dots, X^{r-1}_\bullet), X^r_\bullet)$.

Finally we may define what a cubic hyperresolution is

Definition 0.12. [GNPP88, I, 2.12] Let S be an I-scheme. A *cubic hyperresolution augmented over S* is a $\square_r^+ \times I$-scheme $Z_\bullet$ such that

$$Z_\bullet = \mathrm{rd}(X^1_\bullet, \dots, X^r_\bullet),$$

where

i) $X^1_\bullet$ is a 2-resolution of S,
ii) for $1 \leq n < r$, $X^{n+1}_\bullet$ is a 2-resolution of X^n_1, and
iii) Z_α is smooth for all $\alpha \in \square_r$.

Now that we have defined cubic hyperresolutions, we should note that they exist under reasonable hypotheses

Theorem 0.13. [GNPP88, I, 2.15] *Let S be an I-scheme. Suppose that k is a field of characteristic zero and that I is a finite (bounded) ordered category. Then there exists $Z_\bullet$, a cubic hyperresolution augmented over S such that*

$$\dim Z_\alpha \leq \dim S - |\alpha| + 1, \forall \alpha \in \square_r.$$

Below are some examples of cubic hyperresolutions.

Example 0.14. Let us begin by computing cubic hyperresolutions of curves so let C be a curve. We begin by taking a resolution $\pi : \overline{C} \to C$ (where $\overline{C}$ is just the normalization). Let P be the set of singular points of C; thus P is the discriminant of π. Finally we let E be the (reduced) exceptional set of π, therefore we have the following cartesian square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \overline{C} \arrow[d, "\pi"] \\ P \arrow[r] & C \end{tikzcd}$$

It is clearly already a 2-resolution of C and thus a cubic-hyperresolution of C.

Example 0.15. Let us now compute a cubic hyperresolution of a scheme X whose singular locus is itself a smooth scheme, and whose reduced exceptional set of a strong resolution $\pi : \widetilde{X} \to X$ is smooth (for example, any cone over a smooth variety). As in the previous example, let Σ be the singular locus of X and E the reduced exceptional set of π, Then the cartesian square of reduced schemes

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "\pi"] \\ \Sigma \arrow[r] & X \end{tikzcd}$$

is cartesian, and is in fact is a 2-resolution of X, just as with curves.

The obvious algorithm used to construct cubic hyperresolutions does not construct hyperresolutions in the most efficient or convenient way possible. For example, applying the obvious algorithm to the intersection of three coordinate planes gives us the following.

Example 0.16. Let $X \cup Y \cup Z$ be the three coordinate planes in $\mathbb{A}^3$. In this example we construct a cubic hyperresolution using the obvious algorithm. What makes this construction different, is that the dimension is forced to drop when forming the discriminant of a resolution of a diagram of schemes.

Yet again we begin the algorithm by taking a resolution and the obvious one is $\pi : (X \coprod Y \coprod Z) \to (X \cup Y \cup Z)$. The discriminant is $B = (X \cap Y) \cup (X \cap Z) \cup (Y \cap Z)$, the three coordinate axes. The fiber product making the square below cartesian is simply the exceptional set $E = ((X \cap Y) \cup (X \cap Z)) \coprod ((Y \cap X) \cup (Y \cap Z)) \coprod ((Z \cap X) \cup (Z \cap Y))$, making the following square.

$$\begin{tikzcd}
E = ((X \cap Y) \cup (X \cap Z)) \coprod ((Y \cap X) \cup (Y \cap Z)) \coprod ((Z \cap X) \cup (Z \cap Y)) \arrow[r] \arrow[d, "\phi"] & (X \coprod Y \coprod Z) \arrow[d, "\pi"'] \\
B = (X \cap Y) \cup (X \cap Z) \cup (Y \cap Z) \arrow[r] & X
\end{tikzcd}$$

We now need to take a 2-resolution of the $\underline{2}$-scheme $\phi : E \to B$. We take the obvious resolution that simply separates irreducible components. This gives us $\widetilde{E} \to \widetilde{B}$ mapping to $\phi : E \to B$. The discriminant of $\widetilde{E} \to E$ is a set of three points X_0, Y_0 and Z_0 corresponding to the origins in X, Y and Z respectively. The discriminant of the map $\widetilde{B} \to B$ is simply identified as the origin A_0 of our initial scheme $X \cup Y \cup Z$ (recall B is simply the three axes). The union of that with the images of X_0, Y_0 and Z_0 is again just A_0. The fiber product of the diagram

$$\begin{tikzcd}
 & (\widetilde{E} \to \widetilde{B}) \arrow[d] \\
(\{X_0, Y_0, Z_0\} \to \{A_0\}) \arrow[r] & (\phi : E \to B)
\end{tikzcd}$$

can be viewed as $\{Q_1, \ldots, Q_6\} \to \{P_1, P_2, P_3\}$ where Q_1 and Q_2 are mapped to P_1 and so on (remember E was the disjoint union of the coordinate axes of X, of Y, and of respectively Z, so $\widetilde{E}$ has six components and thus six origins). Thus we have the following diagram

$$\begin{tikzcd}
 & \{Q_1, \ldots, Q_6\} \arrow[rr] \arrow[dl] \arrow[dd] & & \widetilde{E} \arrow[dl] \arrow[dd] \\
\{P_1, P_2, P_3\} \arrow[rr, crossing over] \arrow[dd] & & \widetilde{B} & \\
 & \{X_0, Y_0, Z_0\} \arrow[rr] \arrow[dl] & & E \arrow[dl, "\phi"] \\
\{A_0\} \arrow[rr] & & B \arrow[from=uu, crossing over] &
\end{tikzcd}$$

which we can combine with previous diagrams to construct a cubic hyperresolution.

Example 0.17. Let us do the previous example in a different way. Suppose that S is the union of the three coordinate planes (X, Y, and Z) of $\mathbb{A}^3$. Consider the $\square_2$ or $\square_2^+$ scheme defined by the diagram below (where the dotted arrows are those coming from $\square_2^+$ but not from $\square_2$).

$$\begin{tikzcd}
& X \cap Y \cap Z \arrow[rr] \arrow[dl] \arrow[dd] & & Y \cap Z \arrow[dl] \arrow[dd] \\
X \cap Y \arrow[rr] \arrow[dd] & & Y \arrow[dd, dotted] & \\
& X \cap Z \arrow[rr] \arrow[dl] & & Z \arrow[dl, dotted] \\
X \arrow[rr, dotted] & & X \cup Y \cup Z &
\end{tikzcd}$$

Thus we need to find W_1 and W_2 so that $\mathrm{rd}\{W^1, W^2\}$ gives the diagram above and so that $W^1_\bullet$ is a two-resolution of S. We start by choosing the inclusion $Z \to X \cup Y \cup Z$, which has discriminant $(X \cup Y)$. To complete this square we need in the corner $(X \cup Y) \cap Z$, thus we have the 2-resolution (W^1)

$$\begin{tikzcd}
(X \cup Y) \cap Z \arrow[r] \arrow[d] & Z \arrow[d] \\
X \cup Y \arrow[r] & X \cup Y \cup Z.
\end{tikzcd}$$

Note that the right vertical morphism here is not a resolution of singularities in the usual sense.

Now we must construct a two resolution of the $\underline{2}$-scheme $U = \big(Z \cap (X \cup Y) \to X \cup Y\big)$. Again we will begin with the obvious map from the smooth $\underline{1}$-scheme $V = \big(Y \cap Z \to Y\big)$. Now we need merely construct the discriminant, take fiber products, and hope all the objects are smooth. First we note that the discriminant of $(Y \cap Z) \to (Z \cap (X \cup Y))$ is $X \cap Z$, and so the discriminant of $X \cup Y \to Y$ is X (typically we would also take the union of that with $X \cup Z$, but that is redundant in this case). At the moment, we have the following diagram of $\underline{2}$-schemes

$$\begin{tikzcd}
& \big(Y \cap Z \to Y\big) \arrow[d, dashed] \\
\big(X \cap Z \to X\big) \arrow[r, dashed] & \big(Z \cap (X \cup Y) \to X \cup Y\big)
\end{tikzcd}$$

We take the fiber product (over each element) to obtain $Y \times_{X \cup Y} X = X \cap Y$ and $(Y \cap Z) \times_{Z \cap (X \cup Y)} (X \cap Z) = X \cap Y \cap Z$ (since $k[x,y,z]/(y,z) \otimes_{\frac{k[x,y,z]}{(z,xy)}} k[x,y,z]/(x,z)$ is equal to

$k[x,y,z]/(x,y,z))$. Thus we obtain the diagram below, that we call W^2

$$\begin{tikzcd}
& X \cap Y \cap Z \arrow[rr] \arrow[dl] \arrow[dd] & & Y \cap Z \arrow[dl] \arrow[dd] \\
X \cap Y \arrow[rr] \arrow[dd] & & Y \arrow[dd, dotted] & \\
& X \cap Z \arrow[rr] \arrow[dl] & & (X \cup Y) \cap Z \arrow[dl, dotted] \\
X \arrow[rr, dotted] & & X \cup Y &
\end{tikzcd}$$

Now taking $Z = \mathrm{rd}\{W^1, W^2\}$, we obtain precisely the diagram which we wanted to show was a cubic hyperresolution augmented over $X \cup Y \cup Z \subset \mathbb{A}^3$.

I only mentioned the following briefly in the seminar, but it can be useful.

Remark 0.18. We now want to consider the category of cubic hyperresolutions of schemes (or of I-schemes). Let $\underline{\mathrm{Hrc}}(\underline{\mathrm{Sch}})$ denote the category of hyperresolutions of schemes (or $\underline{\mathrm{Hrc}}(I - \underline{\mathrm{Sch}})$ denote the category of hyperresolutions of I-schemes). There is a (forgetful) functor w from $\underline{\mathrm{Hrc}}(\underline{\mathrm{Sch}})$ back to schemes (or from $\underline{\mathrm{Hrc}}(I - \underline{\mathrm{Sch}})$ back to I-schemes) which obtains the scheme (I-scheme) which the diagram is a hyperresolution of [GNPP88, I, 3.3]. One can also form $\mathrm{Ho}\underline{\mathrm{Hrc}}(\underline{\mathrm{Sch}})$, the category obtained through localization by inverting all morphisms f such that $w(f)$ is an identity morphism on $\underline{\mathrm{Sch}}$. Let $\Sigma_{I-\underline{\mathrm{Sch}}}$ denote the set of morphisms to be inverted. It turns out that w induces an equivalence of categories, $\mathrm{Ho}w$, between $\mathrm{Ho}\underline{\mathrm{Hrc}}(\underline{\mathrm{Sch}})$ and $\underline{\mathrm{Sch}}$ [GNPP88, I. 3.8]. One can say even more: there exists a section of $\mathrm{Ho}w$, that is to say, a functor

$$\eta : \underline{\mathrm{Sch}} \to \mathrm{Ho}\underline{\mathrm{Hrc}}(\underline{\mathrm{Sch}})$$

quasi-inverse to $\mathrm{Ho}w$ satisfying the following properties

(i) $\eta(S) = S$ if S is a smooth scheme

(ii) $\dim \eta(S)_\alpha \leq \dim S - |\alpha| + 1$ for all $\alpha \in \square_r$.

See [GNPP88, I, 3.9]

However, one can go even further:

Corollary 0.19. [GNPP88, I, 3.10] *Let $\mathscr{C}$ be a category. Denote by*

$$w^* : \mathrm{Hom}_{\underline{\mathrm{Cat}}}((I - \underline{\mathrm{Sch}}), \mathscr{C}) \to \mathrm{Hom}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Hrc}}(I - \underline{\mathrm{Sch}}), \mathscr{C})$$

the functor defined by

$$w^*(F) = F \circ w$$

Then w^* induces an equivalence between the categories $\mathrm{Hom}_{\underline{\mathrm{Cat}}}(I - \underline{\mathrm{Sch}}, \mathscr{C})$ and the full subcategory of $\mathrm{Hom}_{\underline{\mathrm{Cat}}}(\underline{\mathrm{Hrc}}(I - \underline{\mathrm{Sch}}), \mathscr{C})$ defined by the functors $G : \underline{\mathrm{Hrc}}(I - \underline{\mathrm{Sch}}) \to \mathscr{C}$ which satisfy the condition below

(DC) *For all morphisms f of $\Sigma_{I-\underline{\mathrm{Sch}}}$, $G(f)$ is an isomorphism of the category $\mathscr{C}$.*

Now let us discuss sheaves on diagrams of schemes, as well as the related notions of push forward and its right derived functors.

Definition 0.20. [GNPP88, I, 5.3-5.4] Let $X_\bullet$ be an I-scheme (or even an I-topological space). We define a *sheaf (or pre-sheaf) of abelian groups $F^\bullet$ on $X_\bullet$* to be the following data:

(i) A sheaf (pre-sheaf) F^i of abelian groups over X_i, for all $i \in \operatorname{Ob} I$, and
(ii) An X_u-morphism of sheaves $F^u : F^i \to (X_u)_* F^j$ for all morphisms $u : i \to j$ of I, also required to be compatible in the obvious way.

Given a map of diagrams of schemes $f_\bullet : X_\bullet \to Y_\bullet$ one can construct a push-forward functor for sheaves on $X_\bullet$.

Definition 0.21. [GNPP88, I, 5.5] Suppose $X_\bullet$ is an I-scheme and $Y_\bullet$ is a J-scheme and that $F^\bullet$ is a sheaf on $X_\bullet$ and $f_\bullet : X_\bullet \to Y_\bullet$ a morphism of diagrams of schemes. We define $(f_\bullet)_* F^\bullet$ in the following way. For each $j \in \operatorname{Ob} J$ we define

$$((f_\bullet)_* F^\bullet)^j = \varprojlim (Y_u)_* (f_{i*} F^i)$$

where the inverse limit traverses all pairs (i, u) where $u : f(i) \to j$ is a morphism of J^{op}.

Remark 0.22. In many applications, J will simply be the category with one object $\{0\}$. In that case one can merely think of the limit as traversing I.

Remark 0.23. One can also define a functor f^*, show that it has a right adjoint and that that adjoint is f_* as defined above [GNPP88, I, 5.5]. We will need f^* so we define it now.

Definition 0.24. [GNPP88, I, Section 5] Let $X_\bullet$ and $Y_\bullet$ be diagrams of topological spaces over I and J respectively, $\phi : I \to J$ a functor, $f_\bullet : X_\bullet \to Y_\bullet$ a ϕ-morphism of topological spaces. If $G^\cdot$ is a sheaf over $Y_\bullet$ with values in a complete category $\mathscr{C}$, one denotes by $f_\bullet^* G^\cdot$ the sheaf over $X_\bullet$ defined by

$$(f_\bullet^* G^\cdot)^i = f_i^*(G^{\phi(i)}),$$

for all $i \in \operatorname{Ob} I$. One obtains in this way a functor

$$f_\bullet^* : \underline{Sheaves}(Y_\bullet, \mathscr{C}) \to \underline{Sheaves}(X_\bullet, \mathscr{C})$$

Given an I-scheme $X_\bullet$, one can define the category of abelian sheaves $\underline{\mathrm{Ab}}$ on $X_\bullet$ and show it has enough injectives, and one can even define the derived category $D^+(X_\bullet, \underline{\mathrm{Ab}})$ by localizing bounded below complexes of sheaves on $X_\bullet$ by the quasi-isomorphisms (those that are quasi-isomorphisms on each $i \in I$). One can also show that $(f_\bullet)_*$ as defined above is left exact so that it has a right derived functor $R(f_\bullet)_*$ [GNPP88, I, 5.8-5.9]. In fact, this functor can be described while using fewer maps than we need to in the regular pushforward. First, however, we need one definition which makes the notation more manageable.

Definition 0.25. [GNPP88, I, 1.5] Let I be a small category and K an I-category (that is a functor $K : I^{\mathrm{op}} \to \underline{\mathrm{Cat}}$). Define $\operatorname{tot}(K)$ to be the category defined below: The objects of $\operatorname{tot}(K)$ are pairs (i, x) such that $i \in \operatorname{Ob} I$ and $x \in \operatorname{Ob} K_i$ (recall $K_i = K(i) \in \operatorname{Ob} \underline{\mathrm{Cat}}$), and the morphisms $(i, x) \mapsto (j, y)$ are pairs (u, a) formed by a morphism $u : i \to j$ of I and by a morphism $a : x \to K_u(y)$ of $K_i \in \operatorname{Ob} \underline{\mathrm{Cat}}$. The composition $(w, c) = (v, b) \circ (u, a)$ of two morphisms $(u, a) : (i, x) \to (j, y)$ and $(v, b) : (j, y) \to (k, z)$ of $\operatorname{tot}(K)$ is defined by $(w, c) = (v \circ u, K_u(b) \circ a)$.

Proposition 0.26. [GNPP88, I, 5.10-5.14] *Suppose that J is a small category and that K is small a J-category, define $I_\bullet = tot(K_\bullet)$ as the category where the morphisms are induced by both those of I and those of J. Let $S_\bullet$ be an J-scheme and let $X_{\bullet\bullet}$ be an $I_\bullet$-scheme augmented* *over $S_{\bullet}$ (that is, we have a map of diagrams of schemes $a_{\bullet} : X_{\bullet\bullet} \to S_{\bullet}$). Note for each $j \in \operatorname{Ob} J$ we have $X_{j\bullet}$ the K_j-scheme (note K_j is a category). Further suppose that $F^{\bullet\bullet}$ is a bounded below complex of abelian sheaves on $X_{\bullet\bullet}$. Then if the components of a_j are denoted by $a_{jk} : X_{jk} \to S_j$, $k \in \operatorname{Ob} K_j$, one has*

$$(Ra_{\bullet *}\mathscr{F}^{\bullet\bullet})^j = R\lim_{\leftarrow K_j} Ra_{jk_*}\mathscr{F}^{jk}, j \in \operatorname{Ob} J$$

Roughly speaking, if we can stratify a diagram $X_{\bullet}$ over another diagram $S_{\bullet}$ then we can use that stratification to simplify the limit (I realize that the notation might suggest we have done almost anything but simplify things).

We can now discuss cohomological descent.

Definition 0.27. Let $K_{\bullet}$ be an I-object of Cat, $\pi : tot(K_{\bullet}) \to I$ the associated projection functor. If $X_{\bullet\bullet}$ is a tot($K_{\bullet}$)-topological space provided with an π-augmentation

$$a_{\bullet} : X_{\bullet\bullet} \to S_{\bullet}$$

over a I-topological space $S_{\bullet}$, we say that $a_{\bullet}$, or by abuse of notation that $X_{\bullet\bullet}$ is of cohomological descent over S_i, if, for all abelian sheaves $\mathscr{F}^{\cdot}$ over $S_{\bullet}$, the morphism of adjunction

$$\mathscr{F}^{\cdot} \to Ra_{\bullet *}a_{\bullet}^{-1}\mathscr{F}^{\cdot}$$

is a quasi-isomorphism.

Remark 0.28. In [PS08], they define something to be of cohomological descent if it satisfies essentially the same property for the constant sheaf $\mathbb{Z}_{\bullet}$. I'm not sure if this is the same condition or not. However, for computing singular cohomology, the weaker definition of [PS08] is fine.

Proposition 0.29. *With the notation of 0.27, $a_{\bullet} : X_{\bullet\bullet} \to S_{\bullet}$ is of cohomological descent over $S_{\bullet}$ if and only if, for all $i \in \operatorname{Ob} I$, the augmentation $a_i : X_{i\bullet} \to S_i$ of the K_i-topological space $X_{i\bullet}$ is of cohomological descent over S_i.*

The proof of this proposition is essentially Proposition 0.26.

Proposition 0.30. *(*[GNPP88, Proposition 6.8], [SGA4, 4.1.2]*). Let*

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

be a cartesian square of morphisms of I-schemes. Suppose the following assumptions

(i) *The morphisms i and i' are closed immersions,*
(ii) *The morphism f is proper, and*
(iii) *The I-scheme Y contains the discriminant (base locus) of f, in other words, f induces an isomorphism of $X'_i - Y'_i$ over $X_i - Y_i$ for all $i \in \operatorname{Ob} I$.*

Under these conditions, if one sets

$$Z_{\bullet} = tot(Y \xleftarrow{g} Y' \xrightarrow{i'} X')$$

and one denotes by $\pi : \square_1 \times I \to I$ the functor of projection, then one has a π-augmentation of diagrams of topological spaces $Z_{\bullet} \to X$ that is of cohomological descent over $X_{\bullet}$.

In effect, according to Proposition 0.29, one can suppose that I is reduced to the category $\underline{1}$. Furthermore (by [GNPP88, Proposition 6.4]), it suffices to prove, for all abelian sheaves $\mathscr{F}$ over X, the acyclicity of simple complex associated to the commutative spare of morphisms of complexes of sheaves over X.

$$\begin{tikzcd} Rh_*h^*\mathscr{F} & Rf_*f^*\mathscr{F} \arrow[l] \\ i_*i^*\mathscr{F} \arrow[u] & \mathscr{F} \arrow[l] \arrow[u] \end{tikzcd}$$

where $h = i \circ g = f \circ i$. However, one has exact sequences of sheaves

$$0 \to j_!j^*\mathscr{F} \to \mathscr{F} \to i_*i^*\mathscr{F} \to 0$$

$$0 \to j'_!j'^*f^*\mathscr{F} \to f^*\mathscr{F} \to i'_*i'^*f^*\mathscr{F} \to 0$$

where $j : X - Y \to X$ and $j' : X' - Y' \to X'$ denote morphisms of inclusion. Given that f is proper, the morphism of adjunction defines a morphism of distinguished triangles of $D^+(X, \underline{Ab})$

$$\begin{tikzcd} J_!j^*\mathscr{F} \arrow[r] & \mathscr{F} \arrow[r] & i_*i^*\mathscr{F} \arrow[r, "+1"] & {} \\ Rf_*j'_!j'^*f^*\mathscr{F} \arrow[r] & Rf_*f^*\mathscr{F} \arrow[r] & Rh_*h^*\mathscr{F} \arrow[r, "+1"] & {} \end{tikzcd}$$

In virtue of (iii), one has $f \circ j' = j$, from which on has an isomorphism

$$Rf_*j'{}_!j'^*f^*\mathscr{F} = j_!j^*\mathscr{F}$$

in $D^+(X, \underline{Ab})$, and the proposition results from the following lemma, which is a variant of the octahedral axiom.

Lemma 0.31. *Let*

$$\begin{tikzcd} \mathscr{F}^{01} \arrow[r, "f^1"] & \mathscr{F}^{11} \\ \mathscr{F}^{00} \arrow[u, "g^0"] \arrow[r, "f^0"'] & \mathscr{F}^{10} \arrow[u, "g^1"'] \end{tikzcd}$$

be a commutative square of morphisms of bounded below complexes of an abelian category. If one sets

$$f^{\cdot} = (f^0, f^1) : s(g^0) \to s(g^1)$$

and

$$g^{\cdot} = (g^0, g^1) : s(f^0) \to s(f^1)$$

the conditions below are equivalent

(i) $f^{\cdot}$ *is a quasi-isomorphism,*
(ii) $g^{\cdot}$ *is a quasi-isomorphism,*
(iii) *the simple complex associated to the* $(\square_1^+)^O$*-complex* $\mathscr{F}^{\cdot}$ *, defined by the diagram above is acyclic.*

Proof. The lemma results from the obvious isomorphism

$$s(g^{\cdot}) = s(\mathscr{F}^{\cdot}) = s(f^{\cdot})$$

because a morphism h of complexes is a quasi-isomorphism if and only if $s(h)$ is acyclic. $\square$

Theorem 0.32. [GNPP88, Theorem 6.9] *Let S be an I-scheme. If $X_{\bullet} \to S$ is a cubic hyperresolution of S, $X_{\bullet}$ is of cohomological descent over S.*

We now discuss simplicial resolutions and how to get one from a cubical hyperresolution.

Definition 0.33. A *semi-simplicial space truncated at level k over X* (in the language of [PS08]) is a collection of topological spaces X_i, $i = -1, \dots, k$ where we set $X_{-1} = X$, together with maps $\epsilon_{ij} : X_i \to X_{i-1}$ for $0 \leq j \leq i \leq k$, as pictured in the diagram below:

$$\begin{tikzcd} X \arrow[r, equal] & X_{-1} & X_0 \arrow[l, "\epsilon_{00}"'] & X_1 \arrow[l, shift right=1, "\epsilon_{10}"'] \arrow[l, shift left=1, "\epsilon_{11}"] & X_2 \arrow[l, shift right=2, "\epsilon_{20}"'] \arrow[l] \arrow[l, shift left=2, "\epsilon_{22}"] & X_3 \arrow[l, shift right=3, "\epsilon_{30}"'] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3, "\epsilon_{33}"] \end{tikzcd}$$

satisfying the relations $\epsilon_{ij} \circ \epsilon_{i+1,j+1} = \epsilon_{ij} \circ \epsilon_{i+1,j}$ for all $j < i$.

Remark 0.34. A "semi-simplicial space truncated at level k over X" is also sometimes called a *simplicial space of level k over X*. See [Ste85] or [Ste87].

We now explain how to construct a semi-simplical space from a cubical one.

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