GENERALIZED DIVISORS AND REFLEXIVE SHEAVES

KARL SCHWEDE

1. Preliminary Commutative Algebra

We introduce the notions of depth and of when a local ring (or module over a local ring) is "S_2". These notions are found in most books on commutative algebra, see for example [Mat89, Section 16] or [Eis95, Section 18]. Another excellent book that is focussed on these ideas is [BH93]. We won't be focussing on the commutative algebra, but one should be aware, at the very least, that this background exists. In particular, we won't really use any of the theory on Cohen-Macaulay rings. However, since one ought to build up the same machinery in order to define S2, we include these definitions as well.

All rings will be assumed to be noetherian.

Definition 1.1. Let $(R, \mathfrak{m})$ be a local ring and let M be a finite R-module (which means finitely generated). An element $r \in R$ is said to be M-regular if $rx \neq 0$ for all $x \in M$, $x \neq 0$ (in other words, r is not a zero divisor on M). A sequence of elements $r_1, \ldots, r_n \in R$ is said to be M-regular if

(i) r_i is a regular $M/(r_1, \ldots, r_{i-1})$ element for all $i \geq 1$ (in particular r_1 is M-regular)
(ii) $(r_1, \ldots, r_n)M \subsetneq M$ (that is, the containment is proper).

Remark 1.2. If condition (i) is satisfied, but condition (ii) is not necessarily satisfied then such a sequence is called *weakly M-regular*.

Remark 1.3. Notice that condition (ii) implies in a regular sequence, all $r_i \in \mathfrak{m} \in R$.

Remark 1.4. One can (and often does) consider the notion of a regular sequence without the hypothesis R is local or that M if finitely generated. However, in that case these notions are not as well behaved (for example, whether or not a sequence is regular depends on the order of the sequence without these hypotheses).

Definition 1.5. Let $(R, \mathfrak{m})$ be a local ring and suppose that M is a finite R-module. We say that M has depth k (or in some books, the *grade of* $\mathfrak{m}$ *on* M *is* k) if the maximal length of an M-regular sequence is k.

We state a few basic properties of depth below. Please see any of the above references for proofs.

Proposition 1.6. *Let $(R, \mathfrak{m})$ be a local ring and suppose that M if a finite R-module. Then*

(a) *If we define a maximal M-regular sequence in the obvious way (it can't be extended to a longer M-regular sequence), then every maximal M-regular sequence has the same length.*
(b) *Modules have a notion of dimension ($\dim M = \dim(A/\operatorname{Ann}_R M)$) and one has $\operatorname{depth} M \leq \dim M$. In particular, R is a finite R-module so we have $\operatorname{depth} R \leq \dim R$.*

(c) *If x is M-regular, then* $\operatorname{depth} M/(xM) = \operatorname{depth} M - 1$ *(here we can view $M/(xM)$ as either an R-module or an $R/(x)$ module).*

Remark 1.7. The ring $(k[x,y,u,v]/((x,y) \cap (u,v)))_{(x,y,u,v)}$ viewed as a module over itself has depth 1 and dimension 2. Geometrically, we just took two planes in $\mathbb{A}^4$ that intersect in a single point, and took the stalk at that intersection point. It is not a bad exercise to check this fact however.

Remark 1.8. Depth can be thought of as a different way to measure the dimension of a module (or ring), which motivates the following definition. One should also note that just like dimension, you can lose depth when you localize. Notice that if we are considering the ring $R = k[x,y]_{(x,y)}$ and we let $M = R/(x)$, then M has depth 1 (you can mod-out by y for example). However, if one localizes at $\mathfrak{p} = (x)$, then $M_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}$ has depth 0 as an $R_{\mathfrak{p}}$-module.

Definition 1.9. A finitely generated module $M \neq 0$ over a local ring $(R, \mathfrak{m})$ is said to be *Cohen-Macaulay* if $\operatorname{depth} M = \dim M$. In particular, by viewing R as a module over itself, we have a notion of what it means for R to be *Cohen-Macaulay*. If M is a finitely generated module over a (not necessarily local) ring R, then we say that M is Cohen-Macaulay if $M_{\mathfrak{m}}$ is Cohen-Macaulay over $R_{\mathfrak{m}}$ for all maximal $\mathfrak{m} \subseteq R$.

Remark 1.10. Being Cohen-Macaulay is preserved by localization, thus M is Cohen-Macaulay if and only if $M_{\mathfrak{p}}$ is Cohen-Macaulay over $R_{\mathfrak{p}}$ for all prime $\mathfrak{p} \subseteq R$. A coherent sheaf is Cohen-Macaulay if all its stalks are and of course a scheme is Cohen-Macaulay if its structure sheaf is. Finally, a regular local ring is clearly Cohen-Macaulay as a module over itself.

Remark 1.11. A large amount of commutative algebra and algebraic geometry is done under the assumption that the rings involved are Cohen-Macaulay. One key fact about them that will come up later for us is that Cohen-Macaulay is the condition under which one can do Serre duality on projective schemes *without* resorting to complicated constructions involving the derived category.

We now define Serre's conditions S_n, which essentially say that the module is Cohen-Macaulay in "low" codimension (the codimension is determined by the n).

Definition 1.12. A module M satisfies *Serre's condition S_n* if

$$\operatorname{depth} M_{\mathfrak{p}} \geq \min(n, \dim R_{\mathfrak{p}})$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. Likewise a coherent sheaf is S_n if all its stalks are.

Remark 1.13. This is a highly non-standard definition of S_2. The more standard approach is to require that

$$\operatorname{depth} M_{\mathfrak{p}} \geq \min(n, \dim M_{\mathfrak{p}}).$$

However, since I am largely following an outline of Hartshorne where he uses this non-standard definition, please keep this in mind. (Or simply imagine that the modules we study satisfy $\dim M_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$, a fairly common situation).

We will be particularly interested in the case where a ring is S_2 (viewed as a module over itself). In particular, if we view R as a module over itself, this says that for primes $\mathfrak{p}$ of height ≤ 2 that $R_{\mathfrak{p}}$ is Cohen-Macaulay, and for primes $\mathfrak{p}$ of height > 2, $R_{\mathfrak{p}}$ has depth at least 2.

Remark 1.14. A very key property of S_2 rings is the following. If R is S_2, then every principal ideal has no embedded primes (see [Mat89, Page 183] or [Mat80, Page 125]). It turns out that normal rings are S_2, which will be very helpful to us.

We also mention a link with sheaves (see [Har77, Chapter III, Exercise 3.5])

Proposition 1.15. *Let X be a Noetherian scheme and let $P \in X$ be a closed point. Then the following are equivalent:*

  (i) $\operatorname{depth} \mathcal{O}_P \geq 2$
  (ii) *if U is any open neighborhood of P, then every section in $\Gamma(U - P, \mathcal{O}_X)$ extends uniquely to a section in $\Gamma(U, \mathcal{O}_X)$.*

Remark 1.16. If X is of finite type over a field, equidimensional (which means that every irreducible component has the same dimension, or equivalently in this context, every closed point a stalk of the same dimension) and has dimension ≥ 2, then X is S_2 if and only if condition (ii) above is satisfied.

We also state the following generalization of one direction. Again, some of the proofs rely on exercises from Hartshorne we haven't done yet. In some sense, the following is why geometers care about S_2 sheaves.

Theorem 1.17. [Har94, Proposition 1.11] *Let X be a noetherian scheme suppose that $\mathscr{F}$ is an S_2 coherent sheaf on X. Let $Y \subset X$ be a closed subset of codimension ≥ 2. Then the restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F})$ is an isomorphism.*

Proof. We first show that the restriction map is injective. Suppose not, then for some section $z \in \Gamma(X, \mathscr{F})$, $\operatorname{Supp} z \in Y$ (see [Har77, Chapter II, Exercise 1.14] for the definition of support). This is preserved after passing to an affine chart, so we consider an ring R and an S_2 module M as well as an element $z \in M$ where we set $I = \operatorname{Ann}_R z$ and we require that I is an ideal of height at least two. Localizing at a minimal prime of I we may assume that R is a local ring and that $\sqrt{\operatorname{Ann}_R z} = \mathfrak{m}$, the maximal ideal of R. This new R still has dimension at least two, by construction and M has depth at least 2. By [Har77, Chapter III, Exercise 3.4], this implies that $\Gamma_{\mathfrak{m}}(M) = H^0_{\mathfrak{m}}(M) = 0$ which means z does not exist (see [Har77, Chapter II, Exercise 5.6] for the definition of $\Gamma_m(M)$). Alternately one could use the previous proposition if $\mathscr{F} = \mathcal{O}_X$ (which amounts to the same thing in this case). Regardless, what we have used the is the fact the module is S_1.

We now follow the proof of [Har94, Proposition 1.11]. To prove surjectivity, it is sufficient to show that every section $s \in \Gamma(X - Y, \mathscr{F})$ extends to a global section of $\mathscr{F}$. Since X is a noetherian topological space, it is sufficient to show that s extends to $\Gamma(X - Y', \mathscr{F})$ where Y' is some proper closed subset of Y. Let y be a generic point of an irreducible component of Y. Let $X_y = \operatorname{Spec} \mathcal{O}_y$. We abuse notation and let $\mathscr{F}$ denote the restriction of $\mathscr{F}$ to X_y. Note that $\dim \mathcal{O}_y \geq 2$.

By some (long) exact sequence using local cohomology [Har77, Chapter III, Exercise 2.3(e)] and again by [Har77, Chapter III, Exercise 3.4], we see that $\Gamma(X_y, \mathscr{F}) \cong \Gamma(X_y - \{y\}, \mathscr{F})$. Now, s induces a section of $\Gamma(X_y - \{y\}, \mathscr{F})$ which extends to a section $s_1 \in \Gamma(X_y, \mathscr{F})$. This is just an element of the stalk $\mathscr{F}_y$ (now undoing our abuse of notation identifying $\mathscr{F}$ with the induced sheaf on X_y). Our goal now is to find a neighborhood where this element and s agree and to use this to shrink Y as above.

There is a an open neighborhood V of y such that $s_2 \in \Gamma(V, \mathscr{F})$ which restricts to s_1. Note that for every point $x \in X_y$, the sections s and s_2 agree by construction. Let $t = s - s_2$ on the open set $(X - Y) \cap V$ and let $Z \subset (X - Y) \cap V$ be the support of t. Thus Z does not meet $X_y - \{y\}$ and so y is not contained in $\overline{Z}$, the closure of Z inside V. Replacing V by $V' = V - Z$, and letting $s_3 = s_2|_{V'}$ we now have V' a new neighborhood of y and the sections $s \in \Gamma(X - Y, \mathscr{F})$ and $s_3 \in \Gamma(V', \mathscr{F})$ which agree on the intersection. Thus since $\mathscr{F}$ is a sheaf, we obtain a section $s' \in \Gamma((X - Y) \cup V', \mathscr{F})$ which restricts to $\mathscr{F}$. But note that $(X - Y) \cup V'$ is strictly larger than $X - Y$ (since it contains y) and so we can let Y' be the complement of $(X - Y) \cup V'$ $\qquad\square$

Corollary 1.18. *Let X be a noetherian scheme suppose that $\mathscr{F}$ is an S_2 coherent sheaf on X. Let $Y \subset X$ be a closed subset of codimension ≥ 2 and set $U = X \backslash Y$. Then if $i : U \to X$ is the natural inclusion, then the natural map $\mathscr{F} \to i_*\mathscr{F}|_U$ is an isomorphism.*

Proof. One can check it on affine charts since $i_*\mathscr{F}|_U$ is quasi-coherent, and on those charts we can apply the previous theorem. $\qquad\square$

Normal schemes are related to S_2 schemes in the following way (originally due to Serre, as are many things).

Proposition 1.19. [Mat89, Theorem 23.8] *A noetherian scheme is normal if and only if it is S_2 and it is R_1 (R_1 means that for every point $P \in X$ such that $\dim \mathcal{O}_{X,P} = 1$, we have that $\mathcal{O}_{X,P}$ is a regular local ring).*

2. Torsion Free and Reflexive Sheaves on Integral Schemes

Remark 2.1. I don't know much about the history reflexive modules and sheaves. Hartshorne has in the past cited a book of Auslander and Bridger, [AB69]. He also cited Bourbaki's commutative algebra, [Bou98], and EGA I, [Gro60].

Throughout this section we always assume that X is quasi-projective over a field (or a noetherian ring). When I say quasi-projective, please choose your favorite field (or noetherian ring) and consider X as quasi-projective over that. This is for the following reason.

FACT: On a quasi-projective variety, every coherent sheaf $\mathscr{F}$ is a quotient of a locally free sheaf.

We discussed this for projective schemes over a noetherian ring, see [Har77, Corollary 5.18]. We can generalize the same result to quasi-projective schemes as follows. Embed X in a projective $\overline{X}$. Then, using [Har77, Exercise 5.15], we can find a coherent sheaf $\overline{\mathscr{F}}$ on $\overline{X}$ that restricts back to $\mathscr{F}$ (this is a very useful exercise). We can then write $\overline{\mathscr{F}}$ as a quotient of locally free sheaves which obviously stay locally free when restricting to X. To prove our main results, none of this is needed, but it makes the statements easier. Note that any scheme is locally quasi-projective over a noetherian ring (itself) thus by making this assumption, we can remove the "locally" hypothesis from many theorems.

Given a coherent sheaf $\mathscr{F}$ on any scheme X, there is the following (dualizing) operation: $\mathscr{F}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. Furthermore there is a natural map from $\mathscr{F}$ to the double-dual, $\mathscr{F} \to (\mathscr{F}^\vee)^\vee$. If this map is an isomorphism, we say that $\mathscr{F}$ *is reflexive* (or more specifically that it is $\mathcal{O}_X$-reflexive). (One can define what it means for a module over a ring to be reflexive in a similar way). Note that if a sheaf is reflexive, it is also coherent (by definition).

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Remark 2.2. If you are unfamiliar with this sort of map, try writing it down first for modules. That is, for a finite R-module M, you want a map

$$M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R).$$

Send $m \in M$ to the map that sends $\phi \in \operatorname{Hom}_R(M, R)$ to $\phi(m)$ (what else could you do?). To get this for schemes, just glue. In order to do this gluing, one has to check that this natural double-duality map is compatible with localization. Note that once one has this (at least if X is integral), one sees that this natural double-dual map is an isomorphism at the generic point of X (where all these modules become modules over a field).

Notice first that any locally free sheaf is reflexive. But there are other reflexive sheaves as well. If one is careful, one can check that $(x, z) \in k[x, y, z]/(xy - z^2)$ corresponds to a reflexive ideal sheaf after taking Spec.

Remark 2.3. If one is unfamiliar with these notions for modules, the following exercise might be helpful. Set $R = k[x, y]$. Which of the following R-modules are reflexive? If a module is not reflexive, compute its double dual $M^{\vee\vee}$.

(a) The ideal (x).
(b) The ideal (x, y).
(c) The module $R/(x, y)$.
(d) The module $R/(x)$.
(e) The ideal $(x^2, xy) = (x, y)^2 \cap (y)$.

There are a few basic facts about reflexive sheaves that should be mentioned. We now limit ourselves to integral schemes which makes dealing with torsion much easier. One can do analogues of the following in more general situations (say for reduced schemes), but the statements become much more involved.

Lemma 2.4. *Suppose that X is a noetherian integral scheme and suppose that $\mathscr{F}$ is a coherent sheaf on X. Then $\mathscr{F}^{\vee}$ is torsion free. (That is, if $U \subset X$ is open and $0 \neq r \in \mathcal{O}_X(U)$ and $0 \neq z \in \mathscr{F}^{\vee}(U)$, then $rz \neq 0$). In particular, a reflexive sheaf is torsion free.*

Proof. First choose $V \subseteq U$ affine such that $z|_V \neq 0$. Note that $r|_V$ is also non-zero since we are dealing with an integral scheme. But then it is enough to show that $rz|_V \neq 0$. Thus we may as well work in the affine case with M a finitely generated R-module with $r \in R$ and $z \in \operatorname{Hom}_R(M, R)$. Suppose that $rz = 0$, this means for every $m \in M$, $r \cdot z(m) = 0$. But since $r \neq 0$, this implies that $z(m) = 0$ for all $m \in M$ which is the same as $z = 0$. $\square$

Note that a torsion-free sheaf is necessarily S_1 (any element makes up a rather short regular sequence).

Lemma 2.5. *Suppose that X is a noetherian integral scheme and that $\mathscr{F}$ is a torsion free coherent sheaf. Then the natural map $\alpha : \mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective.*

Proof. First note that the statement is true if $\mathscr{F}$ is zero so we assume that $\mathscr{F} \neq 0$. The statement is local, so it is harmless to assume that $X = \operatorname{Spec} R$ for some noetherian integral domain R and that $\mathscr{F} = \widetilde{M}$ for some finitely generated module M. Suppose that the natural map $\alpha : M \to M^{\vee\vee}$ is not injective. Thus there is some $0 \neq m \in M$ such that $\alpha(m) = 0$. Let $L = \operatorname{Frac} R$.

Note that $M \otimes_R L$ is a non-zero finitely generated L-vector space. Now consider what happens when we tensor α with L. Because M is finitely generated, we obtain

$$M \otimes_R L \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R) \otimes_R L \cong \operatorname{Hom}_L(\operatorname{Hom}_R(M, R) \otimes_R L, R \otimes_R L) \cong \operatorname{Hom}_L(\operatorname{Hom}_L(M \otimes_R L, L), L).$$

One can then check that this map is the natural map associated to double-dualizing a vector space, which is an isomorphism (again, we're asserting that the double duality morphism localizes well). But then since M is torsion free, $m \otimes 1 \in M \otimes_R L \cong M_{(0)}$ is non-zero, which implies that $\alpha(m)$ cannot be zero. $\qquad\square$

We can generalize the previous lemmas to relate torsion free sheaves to S_1 sheaves.

Proposition 2.6. [Har94, Lemma 1.5] *Let X be a quasi-projective integral scheme and let $\mathscr{F}$ be a coherent sheaf on X. The following are equivalent:*

    (a) *$\mathscr{F}$ is torsion free.*
    (b) *The natural map $\alpha : \mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective.*
    (c) *$\mathscr{F}$ is S_1.*
    (d) *$\mathscr{F}$ is a subsheaf of a coherent locally free sheaf.*

Proof. Lemma 2.4 implies if (b) holds, then so does (a). Lemma 2.5 implies the converse. Of course if d holds, then so does (a). Of course if $\mathscr{F}$ is torsion free, then it is clearly S_1 (you just need a non-zero divisor at every local ring). Conversely, suppose $\mathscr{F}$ is S_1 and consider the kernel K of α. Since X is integral, we see α is injective generically, and thus K is supported in codimension at least 1. If one considers a minimal irreducible component of the support of K, then we see that K at the corresponding stalk has depth zero. But since K injects into $\mathscr{F}$ this contradicts the assumption that $\mathscr{F}$ is S_1.

Finally, we show that (b) implies (d), (as (d) obviously implies (a)). Write $\mathscr{F}^{\vee}$ as a quotient of a locally free sheaf $\mathscr{L}$ (which we can do since X is quasi-projective). Then apply $\mathscr{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X)$. This gives $\mathscr{F}^{\vee\vee}$ (and thus $\mathscr{F}$) as a subsheaf of $\mathscr{L}^{\vee}$ (which is still locally free). $\qquad\square$

Lemma 2.7. [Har80, Proposition 1.1] *A coherent sheaf $\mathscr{F}$ on a quasi-projective integral scheme X is reflexive if and only if it can be included in an exact sequence*

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion free.

Proof. [Har80, Proposition 1.1] Suppose that $\mathscr{F}$ is reflexive. Then we can write an exact sequence

$$\mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F}^{\vee} \to 0,$$

with $\mathscr{L}_0$ and $\mathscr{L}_1$ locally free (since X is quasi-projective).

By applying the left exact contravariant functor $\mathscr{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X)$ we get another exact sequence

$$0 \to \mathscr{F} \to \mathscr{L}_0^{\vee} \to \mathscr{L}_1^{\vee}$$

If we set $\mathscr{G}$ to be the cokernel of $\mathscr{F} \to \mathscr{L}_0^{\vee}$, then $\mathscr{G}$ is isomorphic to a subsheaf of $\mathscr{L}_1^{\vee}$ which is locally free. Thus $\mathscr{G}$ is torsion free.

Conversely, suppose that there is an exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$$

with $\mathscr{E}$ locally free and $\mathscr{G}$ torsion free. But then $\mathscr{F}$ is torsion free so the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective. On the other hand, $\mathscr{E}$ is locally free, and thus reflexive. Thus we have a map $\mathscr{F}^{\vee\vee} \to \mathscr{E}$. At the generic point of X this is injective (since there it is the same as $\mathscr{F} \to \mathscr{E}$). But then it is easy to see that $\mathscr{F}^{\vee\vee} \to \mathscr{E}$ is injective as well since $\mathscr{F}^{\vee\vee}$ is torsion free. Therefore the quotient $\mathscr{F}^{\vee\vee}/\mathscr{F}$ is a subsheaf of $\mathscr{G}$. It is also torsion since it is zero at the generic point of X. But this is impossible since $\mathscr{G}$ is torsion free. $\qquad\square$

Theorem 2.8. *If $\mathscr{F}$ is a coherent sheaf on a noetherian integral scheme X, then $\mathscr{F}^{\vee}$ is reflexive.*

Proof. The statement is local, so we may assume that X is affine. Thus we can write an exact sequence

$$\mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

with the $\mathscr{L}_i$ locally free. Dualizing this as before gives us an exact sequence

$$0 \to \mathscr{F}^{\vee} \to \mathscr{L}_0^{\vee} \to \mathscr{L}_1^{\vee}$$

If one lets $\mathscr{G}$ be the cokernel of $\mathscr{F}^{\vee} \to \mathscr{L}_0^{\vee}$ we again see that $\mathscr{G}$ is torsion free (since it is a subsheaf of $\mathscr{L}_1^{\vee}$). Which is exactly the condition required by the previous lemma in order to prove that $\mathscr{F}^{\vee}$ is reflexive. $\qquad\square$

Corollary 2.9. *Suppose that X is a noetherian integral scheme and that $\mathscr{F}$ is coherent and $\mathscr{G}$ is reflexive, then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ is reflexive.*

Proof. Since $\mathscr{G}$ is reflexive, we can write

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}) \cong \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{G}^{\vee},\mathcal{O}_X)) \cong \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}^{\vee}, \mathcal{O}_X)$$

where the second isomorphism is simply the sheafy-version of Hom/tensor adjointness. $\quad\square$

We now link this notion of reflexive sheaves with the notion of S_2 sheaves.

Theorem 2.10. [Har94, Theorem 1.9] *Suppose that X is a normal integral (not necessarily quasi-projective) noetherian scheme and that $\mathscr{F}$ is a coherent sheaf on X. Then $\mathscr{F}$ is S_2 if and only if $\mathscr{F}$ is reflexive.*

Proof. [Har94, Theorem 1.9] The statement is local, so we may as well assume that X is quasi-projective. First suppose that $\mathscr{F}$ is reflexive. Then there is an exact sequence

$$0 \to \mathscr{F} \to \mathscr{L} \to \mathscr{N} \to 0$$

with $\mathscr{L}$ locally free and $\mathscr{N}$ torsion free. Thus $\mathscr{L}$ satisfies S_2 (this is because X is S_2 since X is normal). Furthermore, $\mathscr{N}$ satisfies S_1 since it is torsion free. We choose a point $P \in X$ and look at the exact sequence on the stalks,

$$0 \to \mathscr{F}_P \to \mathscr{L}_P \to \mathscr{N}_P \to 0.$$

Note if $\dim \mathcal{O}_{X,P} = 1$, then $\mathscr{F}_P$ is a submodule of a free module, so it is S_1 (which is all we would need to check at such a stalk). If $\dim \mathcal{O}_{X,P} \geq 2$, then $\operatorname{depth} \mathscr{L}_P \geq 2$ and $\operatorname{depth} \mathscr{N}_P \geq 1$. Then by considering the long exact sequence of local cohomology (see [Har77, Chapter III, Exercise 3.3]) and [Har77, Chapter III, Exercise 3.4], $\operatorname{depth} \mathscr{F}_P \geq 2$.

Conversely, suppose $\mathscr{F}$ satisfies S_2, then its also S_1. Thus $\alpha : \mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective so let $\mathscr{C}$ be the cokernel. Thus we have an exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}^{\vee\vee} \to \mathscr{C} \to 0.$$

If $\mathscr{C} \neq 0$, let P be a generic point of an irreducible component of the support of $\mathscr{C}$ and consider again the map on stalks. There are two possibilities again.

First if $\dim \mathcal{O}_{X,P} = 1$, then $\mathcal{O}_{X,P}$ is normal, and so it is a discrete valuation ring and in particular a PID. But then $\mathscr{F}_P$ is free (since it is torsion free) and so $\mathscr{F}_P$ is already reflexive and α_P is an isomorphism and $\mathscr{C}_P = 0$.

Now suppose that $\dim \mathcal{O}_{X,P} \geq 2$ and consider the sequence

$$0 \to \mathscr{F}_P \to \mathscr{F}_P^{\vee\vee} \to \mathscr{C}_P \to 0.$$

Here $\mathscr{F}_P$ has depth ≥ 2 as does $\mathscr{F}_P^{\vee\vee}$ (since it is reflexive). Furthermore, $\mathscr{C}_P$ has depth zero since it is supported at P. If one takes the local cohomology sequence again (at the local ring $\mathcal{O}_{X,P}$, see [Har77, Chapter III, Exercise 3.3]) we see that this is impossible. $\qquad\square$

Corollary 2.11. *Let X be a integral, normal (not necessarily quasi-projective) scheme and suppose that $\mathscr{F}$ is a reflexive sheaf on X (defined as above). Let $Y \subset X$ be a closed subset of codimension ≥ 2 and set $U = X \backslash Y$. Then if $i : U \to X$ is the natural inclusion, then the natural map $\mathscr{F} \to i_* \mathscr{F}|_U$ is an isomorphism.*

One nice property of reflexive sheaves is that given a finite morphism of normal integral schemes, then the pushforward of a reflexive sheaf is reflexive, [Har80].

We also have the following fact related, which we will find useful.

Proposition 2.12. *Suppose that $\mathscr{F}$ is a reflexive sheaf on $U \subseteq X$ (where X is as above) where $X - U$ is codimension two. Let us denote by $i : U \to X$ the inclusion. Then $i_* \mathscr{F}$ is a reflexive sheaf on X*

Proof. Certainly $(i_* \mathscr{F})^{\vee\vee}$ is reflexive, so the natural map

$$(i_* \mathscr{F})^{\vee\vee} \to i_* ((i_* \mathscr{F})^{\vee\vee}|_U)$$

is an isomorphism However, $(i_* \mathscr{F})^{\vee\vee}|_U$ is clearly the same as $\mathscr{F}$, since $\mathscr{F}$ was reflexive on U. Therefore $(i_* \mathscr{F})^{\vee\vee} \cong i_* \mathscr{F}$ and we are done. $\qquad\square$

3. Divisors and Reflexive Sheaves

I don't know of a good reference for what follows, although certainly the basics have been outlined in various places, see for example [Kc92, Chapter 16] or [Fer01]. Of course, the works of Hartshorne I have previously cited ([Har80], [Har94] and [Har07]) also have a certain amount of this worked out but not necessarily phrased in this language and also worked out in greater generality. Also see [Bar77].

Let X be a normal integral separated noetherian scheme of finite type over a field (or, instead of the finite type condition, you may assume that there is a cover of X by affine charts whose corresponding rings are excellent, see [Mat80]). By a Weil divisor on X, we mean a formal sum of integral codimension 1 subschemes (prime divisors). Just like in the regular case, each prime divisor D corresponds to some discrete valuation v_D of the fraction field of X (although the reverse direction is not true). This is because the stalk at the generic point of a prime divisor is still a regular ring (since normal rings are R_1). Also like before, we can form $\mathrm{Div}(X)$, the group of divisors on X.

Remark 3.1. One key fact that we observe is that since X is normal it is R_1. This also implies that, since the non-regular locus is closed, that the non-regular locus is a closed subset of codimension 2 (again using the finite type over a field condition, or the "excellent"

condition). We will often denote its complement by U and then note that the results of the previous sections apply in terms of extending S_2 sheaves on U to X via the pushforward map. Furthermore, just as before, $\text{Div}(X) \cong \text{Div}(U)$.

Definition 3.2. Choose $f \in K(X)$, $f \neq 0$. We define the *principal divisor* $\text{div}(f)$ as in the regular case: $\text{div}(f) = \Sigma_i v_{D_i}(f) D_i$. Likewise, we say that two Weil divisors D_1 and D_2 are *linearly equivalent*, if $D_1 - D_2$ is principal.

Definition 3.3. Given a divisor D, we define $\mathcal{O}_X(D)$ be the sheaf associated to the following rule:
$$\Gamma(V, \mathcal{O}_X(D)) = \{f \in K(X)|\operatorname{div}(f)|_V + D|_V \geq 0\}$$

Proposition 3.4. *Suppose that D is a prime divisor, then $\mathcal{O}_X(-D) = \mathscr{I}_D$ and furthermore, if D is any divisor, then $\mathcal{O}_X(D)$ is reflexive.*

Proof. We first show the equality. The object defined above is clearly a sheaf. We will prove the equality of the sheaves in the setting where U is affine. Then $\Gamma(U, \mathcal{O}_X(D))$ is just the functions in $\mathcal{O}_X$ which vanish to order at least 1 along D, in other words the ideal of D.

We now want to show that this sheaf is reflexive (or equivalently, that it is S_2). First notice that clearly if U is the regular locus of X, then $\Gamma(V \cap U, \mathcal{O}_X(D)) \cong \Gamma(V, \mathcal{O}_X(D)$ for any open set V. This is because $V \cap U = U \backslash \{\text{non-regular locus}\}$, the non-regular locus is codimension 2, and the sections of $\mathcal{O}_X(D)$ obviously do not change when removing a codimension 2 subset. This implies that the natural map $\mathcal{O}_X(D) \to i_* \mathcal{O}_X(D)|_U$ is an isomorphism, but then we notice that $\mathcal{O}_X(D)|_U$ is reflexive (since it is invertible) and thus, by corollary 2.11, $\mathcal{O}_X(D)$ is also reflexive.

Alternately, if one wants to show (directly) that it is S_2, then the statement easily from [Har77, Chapter III, Exercise 3.5] by looking at various stalks, taking the Spec of the corresponding and asking whether one can extend the sections from the punctured spectrum. $\square$

We prove some other simple facts about divisors and reflexive sheaves below.

Proposition 3.5. [Har80, Proposition 1.9] *Every rank one reflexive sheaf $\mathscr{F}$ on a regular scheme X is invertible.*

I won't prove this but I will describe a sketch of a proof. One first shows that there exists a closed set Z of codimension 2 such that $\mathscr{F}|_{X \backslash Z}$ is free. On the other hand, since X is regular, one can show that $\text{Pic}(X) \cong \text{Pic}(X \backslash Z)$. (Here $\text{Pic}(X)$ is defined to be the group of isomorphism classes of invertible sheaves on X).

Proposition 3.6. *Every rank one reflexive sheaf $\mathscr{F}$ on a normal scheme X embeds as a subsheaf of $\mathscr{K}(X)$.*

Proof. Let Z denote the non-regular locus of X, which by assumption is codimension 2 since X is normal. Set $U = X \backslash Z$ and let $i : U \to X$ be the inclusion, then $\mathscr{F}|_U$ is invertible. On the other hand, there is a map $\mathscr{F} \to i_* \mathscr{F}|_U$ and we have a diagram

$$
\begin{array}{ccc}
\mathscr{F} & & \mathscr{K}(X) \\
\downarrow & & \downarrow \\
i_* \mathscr{F}|_U & \longrightarrow & i_* \mathscr{K}(X)|_U
\end{array}
$$

Since $\mathscr{F}$ is reflexive, the first vertical arrow is an isomorphism. Since $\mathscr{K}(X)$ is just a constant sheaf on an irreducible topological space, the second vertical arrow is also an isomorphism. $\qquad \square$

Proposition 3.7. *Any reflexive rank 1 subsheaf of $\mathscr{K}(X)$ is $\mathcal{O}_X(D)$ for some (uniquely determined) divisor D.*

Proof. Let U be the regular locus of X. We can consider the following commutative diagram:

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{K}(X) \\ \downarrow & & \downarrow \\ i_*\mathscr{F}|_U & \longrightarrow & i_*\mathscr{K}(X)|_U \end{array}$$

The fact that this diagram is commutative is the same as saying that the map $\mathscr{M} \to i_*\mathscr{M}|_U$ is natural (I'll leave this as an exercise). As before, we see that both vertical arrows are isomorphisms. Furthermore, $\mathscr{F}|_U$ corresponds to divisor D on U and so $\mathscr{F}|_U = \mathcal{O}_U(D)$. If we extend D to X, we obtain some divisor D' on X. But then $\mathcal{O}_X(D') \cong i_*\mathscr{F}|_U \cong \mathscr{F}$ since they agree on U, and the inclusion into $\mathscr{K}(X)$ is clearly the same as the one defined above. $\qquad \square$

In fact, just as before, there is a bijective correspondence between rank 1 reflexive subsheaves of $\mathscr{K}(X)$ and divisors D on X. Even more, we can characterize the divisor class group ($\mathrm{Div}(X)$ modulu principal divisors) using reflexive sheaves (see below).

Definition 3.8. We call a Weil divisor D *Cartier* if $\mathcal{O}_X(D)$ is an invertible sheaf.

Proposition 3.9. *Let $f \in K(X)$ be non-zero where X is as above. Then $\mathrm{div}(f)$ is a Cartier divisor.*

Proof. We know that on the regular locus $U \subseteq X$, $\mathcal{O}_U(\mathrm{div}(f)|_U) = \frac{1}{f}\mathcal{O}_U$, but we also note that $\frac{1}{f}\mathcal{O}_X$ is reflexive (since abstractly it is isomorphic to $\mathcal{O}_X$) but likewise, so is $\mathcal{O}_X(\mathrm{div}(f))$. So it is easy to see that they are the same subsheaf of $\mathscr{K}(X)$ (one can use pushforward tricks to obtain a map, or one explicitly write down a map between them which is an isomorphism on U). $\qquad \square$

Note the following observation.

Lemma 3.10. *If D is a Weil divisor and E is a Cartier divisor, then $\mathcal{O}_X(D + E) \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(E)$*

Proof. The statement is local, and $\mathcal{O}_X(E)$ is locally free of rank one. $\qquad \square$

We can generalize it to a specific case as follows:

Proposition 3.11. *Suppose that D_1 and D_2 are two Weil divisors, then D_1 and D_2 are linearly equivalent if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ (abstractly).*

Proof. Certainly if D_1 and D_2 are linearly equivalent, then $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2 + \mathrm{div}(f))$ for some non-zero $f \in K(X)$. But then we are done by the previous lemma and proposition.

Conversely if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$, then on the regular locus $U \subseteq X$, we can find an element $f \in K(X)$ such that $(D_1 - D_2)|_U = \mathrm{div}(f)|_U$. But then this extends to the non-regular locus as well. $\qquad \square$

As before, we define a Weil divisor $D = \Sigma_i a_i D_i$ to be *effective*, if $a_i \geq 0$.

Note that if D is effective, then $\mathcal{O}_X \subset \mathcal{O}_X(D)$ (inside $\mathcal{K}(X)$) and conversely. Furthermore, given a rank one reflexive sheaf $\mathscr{F}$, an injective (equivalently non-zero) map

$$\mathcal{O}_X \to \mathscr{F}$$

uniquely determines $\mathscr{F}$ as a subsheaf of $\mathcal{K}(X)$. One way to see this is to note that one obtains such an inclusion on the regular locus $U \subseteq X$, which we can then pushforward onto all of X as before. Therefore, just as we discussed before, the global sections of $\mathscr{F}$ determine effective divisors D such that $\mathscr{F} \cong \mathcal{O}_X(D)$ (in other words, they determine to effective divisors D in the linear equivalence class of $\mathscr{F}$).

One thing to be careful of. Even though every global section makes $\mathscr{F}$ into a subsheaf of $\mathcal{K}(X)$, there can be two distinct global sections which turn $\mathscr{F}$ into the same subsheaf (just as (x) and $(-2x)$ are the same ideal inside $\mathbb{C}[x]$). These facts are summarized (and expanded upon) in the proposition below (compare with [Har77, Chapter II, Proposition 7.7]

Proposition 3.12. *Suppose that X is as above and that $\mathscr{F}$ is a rank 1 reflexive sheaf on X.*

(a) *To every non-zero global section $s \in \Gamma(X, \mathscr{F})$, we can associate an effective divisor D on X.*

(b) *For every effective divisor D such that $\mathcal{O}_X(D) \cong \mathscr{F}$, there is a section $s \in \Gamma(X, \mathscr{F})$ such that s corresponds to D.*

(c) *Two non-zero global sections s_1, s_2 determine the same divisor if and only if there is a unit u in $\Gamma(X, \mathcal{O}_X)$ such that $s_1 = us_2$.*

Proof. To prove (a), we give a different argument than the on in the paragraph above. Note that since $\mathscr{F}$ is torsion free and rank 1, we have $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{K}(X) \cong \mathcal{K}(X)$ (just as in the locally free case). Therefore we have a commutative diagram.

$$
\begin{array}{ccc}
\mathcal{O}_X & \longrightarrow & \mathscr{F} \\
\downarrow & & \beta \downarrow \\
\mathcal{K}(X) & \xrightarrow{\phi} & \mathcal{K}(X)
\end{array}
$$

Note that ϕ is an isomorphism, and so we embed $\mathscr{F}$ into $\mathcal{K}(X)$ via $\phi^{-1} \circ \beta$. This embedding has an associated divisor by 3.7.

To prove (b), simply take the image of 1 in the natural map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X(D)) \cong \mathscr{F}$ and then use the above.

Finally, to prove (c), suppose that s_1 and s_2 are two global sections of $\mathscr{F}$ which determine the same divisor (in other words, which determine the same subsheaf of $\mathcal{K}(X)$). This gives us two embeddings $\beta_i : \mathscr{F} \to \mathcal{K}(X)$ which have the same image. The two sections s_i each have image to $1 \in K(X)$ in the embedding β_i. Let u be $\frac{1}{\beta_1(s_2)}$ (thus u is the reciprocal of the β_1-image of s_2). It then follows that $(\times u) \circ \beta_1 : \mathscr{F} \to \mathcal{K}(X)$ is an embedding of $\mathscr{F}$ into $\mathcal{K}(X)$ that sends s_2 to 1 (here $\times u$ simply means the map of sheaves induced by multiplying by u), and thus it must be β_2. Therefore we see that $u\beta_1(\mathscr{F}) = (\times u)(\beta_1(\mathscr{F}))$ must equal $\beta_2(\mathscr{F}) = \beta_1(\mathscr{F})$.

Now consider an affine open cover $\{V_i\}$ of $U = X \backslash$ non-regular locus of X where $\mathscr{F}|_{V_i}$ is free. Thus we still see that $u\Gamma(V_i, \beta_1(\mathscr{F})) = \Gamma(V_i, \beta_1(\mathscr{F}))$. But this implies that u is a unit in $\Gamma(V_i, \mathcal{O}_X)$ since abstractly, $\Gamma(V_i, \beta_1(\mathscr{F})) \cong \Gamma(V_i, \mathcal{O}_X)$. Therefore, u can also be viewed

as a unit in $\Gamma(U, \mathcal{O}_X)$. But X is normal and so in particular X is S_2 , which implies that $\Gamma(U, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)$ and we see that u is a unit (and particularly an element) in $\Gamma(X, \mathcal{O}_X)$ as well (since all the restriction maps on $\mathcal{O}_X$ are injective and we can identify any section with a unique element of $K(X)$). Finally we note that $\beta_1(us_2) = u\beta_1(s_2) = 1 = \beta_1(s_1)$, but β_1 is injective and the proof is complete.

The converse of (c) is trivial since a unit clearly restricts to units in other section rings. $\square$

Finally we discuss adding divisors and the corresponding tensor operations on sheaves.

Proposition 3.13. *Let D_1 and D_2 be two divisors on X (where X is as above). Then we have the following facts:*

(a) $\mathcal{O}_X(D_1 + D_2) \cong (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee}$
(b) $\mathcal{O}_X(-D_1) \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D_1), \mathcal{O}_X)$
(c) $\mathcal{O}_X(D_1 - D_2) \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D_2), \mathcal{O}_X(D_1)) \cong (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(-D_2))^{\vee\vee}$

Proof. These facts hold on the regular locus U. Simply pushforward. $\square$

Thus one can turn the set of (isomorphism classes of) rank 1 reflexive sheaves into a group as follows. Two add two sheaves, simply tensor them together and then double-dualize (apply $\vee$ twice). To invert a sheaf, simply dualize. $\mathcal{O}_X$ is the identity. This group is clearly isomorphic to the divisor class group by the previous results.

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