MATH 217-4, EXTRA CREDIT #5

The purpose of this extra credit assignment is to construct new vector spaces from existing vector spaces. We have already done this in certain situations. For example, given a vector space V and a set S of vectors of V, then $\mathrm{Span}S$ is a vector space itself, and a subspace of S.

Exercise 0.1. In the context above, prove that $\mathrm{Span}(S)$ is the smallest subspace of V that also contains S. (1 point)

Exercise 0.2. Suppose V is a vector space and $L \subset V$ is a subspace. Prove that $\mathrm{Span}(L) = L$. (1 point)

We are going to give two other ways to construct vector spaces. The first is the cartesian product.

Definition 0.3. Let V and W be real-vector spaces. Define $V \times W$ to be the collection of all pairs (v, w) where $v \in V$ (ie, v is a "vector" in V) and $w \in W$ (ie, w is a "vector" in W). We define addition of pairs to be

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2).$$

We define multiplication of such a pair by a scalar $r \in \mathbb{R}$ to be

$$r(v, w) = (rv, rw).$$

Exercise 0.4. Prove that $V \times W$, with the addition and multiplication outlined above, is indeed a vector space. Hint: there are a lot of properties to check (2 points)

That should have been fairly straightforward, so now lets do something a bit more challenging.

Given a linear transformation $T : V \to W$ of vector spaces, we already know that $\ker T$ (ie, the kernel of T) is always a subspace of V, and thus a vector space in its own right. We are going to construct a vector space which in some ways is dual to $\ker T$.

We are going to start by looking at the situation of W a vector space and $L \subset W$ a subspace. For each $w \in W$, we consider the set $w + L = \{w + l \text{ where } l \in L\}$ (we thought about sets like this back in the first extra credit assignment).

Exercise 0.5. Prove that $w + L = w' + L$ if and only if $w - w' \in L$ (the last statement means that $w - w'$ is a "vector" in the subspace L). (1 point)

We want to prove that the collection of all these $(w + L)$'s (where we throw out duplicate sets) is itself a vector space. We'll denote the set of all $(w + L)$'s (throwing out duplicates) as W/L. To do this, we have to define addition, scalar multiplication, and show they are *well defined*. I'll explain what this means momentarily.

We begin by defining addition. Give $w + L$ and $u + L$, we define their sum as follows,

$$(w + L) + (u + L) = (w + u) + L$$

In this case, being well defined means that if we are adding $w + L$ to $u + L$, then if $w + L = w' + L$, it doesn't matter whether we add $w + L$ or $w' + L$, we get the same output.

Exercise 0.6. Prove that the addition defined above is well defined. That is, prove that if $w + L = w' + L$, then for every $u + L$ we have

$$(w + L) + (u + L) = (w' + L) + (u + L).$$

Hint: Use 0.5. (1 point)

We now define scalar multiplication. Given $w + L$ an element of W/L, and $r \in \mathbb{R}$. We define scalar multiplication as follows:

$$r(w + L) = rw + L.$$

Of course, now we have to check that this is "well defined".

Exercise 0.7. Prove that the scalar multiplication defined above is well defined. That is, prove that if $w + L = w' + L$, then for every $r \in \mathbb{R}$ we have

$$r(w + L) = r(w' + L).$$

Hint: Use 0.5. (1 point)

Now we have given W/L addition and scalar multiplication, we need to finish checking that it is a vector space.

Exercise 0.8. Prove that W/L, with the addition and multiplication defined above, is a vector space. Hint: there are a lot of properties to check. Pay particular attention to what elements w make $w + L$ the "zero vector" of W/L (2 points)

This construction, W/L, is called a quotient vector space. When talking about this space out loud, it is called "W modulo L", or simply "W mod L". We would like to mention one final property of this construction.

Exercise 0.9. Given W and L as above, we would like to define a function $\phi : W \to W/L$ as follows

$$\phi(w) = w + L.$$

Prove that ϕ defined this way is an onto (aka surjective) linear transformation. (1 point)

Exercise 0.10. With the notation of the previous exercise, find the kernel of the linear transformation $\phi : W \to W/L$. Then use it to explain why the statement, "Every subspace is the kernel of some linear transformation", is true. (1 point)

I would like to show one place where this construction comes up very often. Just like we started with, suppose that $T : V \to W$ is a linear transformation. First, recall that the range (aka image) of T is itself a subspace of W, and we denote it by $\operatorname{im} T \subset W$. The vector space $W/(\operatorname{im} T)$ is called the *cokernel of* T and is denoted by $\operatorname{coker} T$.

This extra credit assignment will conclude with a famous theorem called "the first isomorphism theorem". Here's the setup.

Suppose that A and B are vector spaces and that $f : A \to B$ is a linear transformation of vector spaces. There is a natural onto (aka surjective) linear transformation $\phi : A \to A/(\ker f)$ (coming from 0.9). Define a function $\psi : A/(\ker f) \to B$ by $\psi(a + (\ker f)) = f(a)$. We wish to show that this map is well defined.

Exercise 0.11. Prove that ψ is well defined. That is, suppose that $a + (\ker f) = a' + (\ker f)$. Prove that $\psi(a + (\ker f)) = \psi(a' + (\ker f))$. Hint: You might want to first prove that $f(a) = f(a')$ whenever $a - a' \in \ker f$. (1 point)

Exercise 0.12. Prove that the map ψ defined above is a one-to-one (aka injective) linear transformation. (1 point)

Exercise 0.13. Prove that the linear transformation $f : A \to B$ can be written as a composition of linear transformations $\phi : A \to A/(\ker f)$ and $\psi : A/(\ker f) \to B$. In short, prove that $f = \psi \circ \phi$. Explain why this shows that every linear transformation can be written as a composition of a surjective linear transformation, followed by an injective linear transformation. (1 point).

Exercise 0.14. Explain why (in this setup) there is always a natural bijective linear transformation (aka an isomorphism) $A/(\ker f) \to \operatorname{im} f$. Hint: what is the range of ψ? (1 point)

The previous four exercises make up "the first isomorphism theorem for vector spaces". These observations are fundamental to an extremely large chunk of modern mathematics.