INTRODUCTION TO DIFFERENTIAL EQUATIONS

MATH 186–1

1. Ordinary differential equations

We work with real numbers in this worksheet.

Definition 1.1. Fix x to be a variable, and $y : [a, b] \to \mathbb{R}$ to be an unknown function (of x).

An *ordinary differential equation* is an equation which relates derivatives of y with x and/or y and/or other derivatives of y.

Example 1.2. The following are examples of differential equations.

(1) $y' = x^2 + x$
(2) $y' = 2x\cos(x^2)$
(3) $y' = 0$
(4) $y' = y$
(5) $y' = xy$
(6) $y' = x^2y + y$
(7) $y' = y + x + 1$
(8) $y'' + y' + 2y = 0$
(9) $y^{(3)} = -y'$

Definition 1.3. The *order* of a differential equation is the degree of the highest derivative of y that appears in the differential equation.

1. Find the orders of the differential equations from Example 1.2.

Solution: (1) = 1, (2) = 1, (3) = 1, (4) = 1, (5) = 1, (6) = 1, (7) = 1, (8) = 2, (9) = 3

Definition 1.4. To *solve* a differential, you need to find a function y that satisfies the equation.

2. Find *all* solutions to the differential equations (1), (2) and (3) from Example 1.2. Why can't we do something similar for the other examples?

Solution:
(1) $y' = x^2 + x$ so $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$ works for any constant C.
(2) $y' = 2x\cos(x)$ so $y = \sin(x^2) + C$ works for any constant C.
(3) $y' = 0$ so $y = C$ works for any constant C.
This doesn't work for the other problems because they aren't $y' =$ a function of x.

We'll discuss our first method for finding a solution to a differential equation.

Definition 1.5. A first order differential equation is called separable if it can be written in the form $y' = f(y)g(x)$.

3. Which of the equations from Example 1.2 are separable?

Solution: Equations (1), (2), (3), (4), (5), (6) are all separable.

If a differential equation is separable, you can rewrite it in the form $\frac{1}{f(y)}y' = g(x)$ (at least as long as $f(y)$ is not zero). Define a new function H to be a antiderivative of $\frac{1}{f(x)}$. Then $(H(y))' = \frac{1}{f(y)}y'$ by the chain rule. If we further let G be any antiderivative of $g(x)$, then we know that

$$(H(y))' = G'(x)$$

and so $H(y) = G(x) + C$ where C is some constant (remember, two differentiable functions with the same derivative differ up to a constant).

Now you have an implicit equation relating y and x. You can hopefully solve for y.

Example 1.6. Consider the differential equation $y' = y^2$, it is separable with $f(y) = y^2$ and $g(x) = 1$. Write $\frac{1}{y^2}y' = 1$. We want $H(x)$ to be an anti-derivative of $\frac{1}{x^2}$ (we assume $x \neq 0$), $H(x) = \frac{-1}{x}$ certainly works. We set $G(x)$ to be an anti-derivative of 1, $G(x) = x$ works. Then we know that $H(y) = G(x) + C$. Thus $\frac{-1}{y} = x + C$ and so $y = \frac{-1}{x+C}$.

To check our work, we plug this solution back into our original equation. $y' = y^2$. We get

$$y'(x) = (-1)\frac{-1}{(x+C)^2} = \frac{1}{(x+C)^2} = \frac{(-1)^2}{(x+C)^2} = y^2(x)$$

as desired.

4. Find a solution to each of the differential equations you said were separable (that you haven't already solved before). Check your solutions!

Solution: We need to solve (4), (5) and (6).

(4) $y'\frac{1}{y} = 1$. So $f(y) = \frac{1}{y}$. An antiderivative is $H(y) = \ln(y)$. G is an antiderivative of 1 so $G(x) = x$ works. Write $\ln(y) = x + C$ and so $y = e^{x+C} = ke^x$ (for some non-zero constant k, of course, if $k = 0$,it's easy to check that $y = 0$ is also a solution). To check our solution, notice that $y' = (ke^x)' = ke^x = y$.

(5) $y'\frac{1}{y} = x$. Just as in (4), $H(y) = \ln(y)$ but now $G(x) = \frac{1}{2}x^2$. Set $\ln(y) = \frac{1}{2}x^2 + C$ and so $y = e^{\frac{1}{2}x^2+C} = ke^{x^2/2}$ for some constant k. To check our solution, notice that $y' = ke^{x^2/2}(x^2/2)' = xke^{x^2/2} = xy$ as desired.

(6) $y' = (x^2+1)y$ and thus $y'\frac{1}{y} = (x^2+1)$. Again, $H(y) = \ln(y)$. Now, $g(x) = x^2 + 1$ so $G(x) = \frac{1}{3}x^3 + x$. Thus $\ln(y) = \frac{1}{3}x^3 + x + C$ and so (now skipping a step) $y = ke^{\frac{1}{3}x^3+x}$ for some constant k. To check our solution, notice that

$$y' = ke^{\frac{1}{3}x^3+x}(\frac{1}{3}x^3 + x)' = ke^{\frac{1}{3}x^3+x}(x^2+1) = x^2y + y$$

as desired.

Another way to find solutions to differential equations is to find power series that solve the differential equations. For example, consider again the differential equation $y' = y$. If we imagine that y can be written as a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then we know that

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and so $\frac{1}{(n+1)} a_n = a_{n+1}$ for every n. This is a *recursive formula* for the coefficients. In particular, if we fix any a_0, then

$$a_1 = a_0 \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2}a_0$$

$$a_3 = \frac{1}{3}a_2 = \frac{1}{3!}a_0 \qquad a_4 = \frac{1}{4}a_3 = \frac{1}{4!}a_0$$

$$\dots \qquad a_n = \frac{1}{n!}a_0$$

So a solution is $y(x) = \sum_{n=0}^{\infty} a_0 \frac{1}{n!} x^n$ for any a_0. (It is easy to check that this converges)

5. Show that the power series solution written above is the same solution as the one you found when you did problem **4.**

Solution: We only need to check (4) because that is the one that's worked out above. For (4), we found that $y = ke^x$ is a solution (where k is an arbitrary constant). Above, we found that $\sum_{n=0}^{\infty} a_0 \frac{1}{n!} x^n$ is also a solution where a_0 is an arbitrary constant, but

$$\sum_{n=0}^{\infty} a_0 \frac{1}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = a_0 e^x.$$

So we simply can set $k = a_0$ and note the solutions are the same.

6. Find a power series solution to (5) and (7) from Example 1.2. What are the radii over convergence?

Solution: (5) $y' = xy$. So suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a solution. Then

$$\begin{array}{rcccccc} y' = & a_1x^0+ & (2)a_2x^1+ & (3)a_3x^2+ & (4)a_4x^3+ & (5)a_5x^4+ & \dots \\ = xy = & & a_0x^1+ & a_1x^2+ & a_2x^3+ & a_3x^4+ & \dots \end{array}$$

Thus $a_1 = 0$, a_0 is arbitrary, and in general, $na_n = a_{n-2}$ for $n \geq 2$. Therefore, $a_n = 0$ for n odd since $3a_3 = a_1 = 0$, and $5a_5 = a_3 = 0$, etc. Furthermore

$$\begin{array}{rcl} 2a_2 = a_0 & \text{so} & a_2 = \frac{1}{2}a_0 \\ 4a_4 = a_2 = \frac{1}{2}a_0 & \text{so} & a_4 = \frac{1}{4}\frac{1}{2}a_0 \\ 6a_6 = a_4 = \frac{1}{4}\frac{1}{2}a_0 & \text{so} & a_6 = \frac{1}{6}\frac{1}{4}\frac{1}{2}a_0 \\ 8a_8 = a_6 = \frac{1}{6}\frac{1}{4}\frac{1}{2}a_0 & \text{so} & a_8 = \frac{1}{8}\frac{1}{6}\frac{1}{4}\frac{1}{2}a_0 \\ \dots & & \dots \end{array}$$

And in general, $a_{2n} = \frac{1}{2n}\frac{1}{2(n-1)}\frac{1}{2(n-2)} \cdots \frac{1}{2}a_0 = \frac{1}{2^n n!}a_0$. Thus $a_0 \sum_{n=0}^{\infty} \frac{1}{2^n n!} x^{2n}$ is a solution. One can easily check using the ratio test that the radius of convergence is infinity.

Solution: (6) Suppose that $y = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots$ is a solution. Then

$$\begin{array}{rcccccc} y' = & a_1x^0+ & (2)a_2x^1+ & (3)a_3x^2+ & (4)a_4x^3+ & (5)a_5x^4+ & \dots \\ = y + x + 1 = & (a_0+1)x^0+ & (a_1+1)x^1+ & a_2x^2+ & a_3x^3+ & a_4x^4+ & \dots \end{array}$$

So $a_1 = a_0 + 1$, thus $a_2 = \frac{1}{2}(a_1 + 1) = \frac{1}{2}(a_0 + 2)$ and further on $a_n = \frac{1}{n}a_{n-1} = \frac{1}{n}\frac{1}{(n-1)} \cdots \frac{1}{2}(a_0 + 2)$. So the general solution is

$$y = a_0x^0 + (a_0+1)x^1 + \frac{1}{2}(a_0+2)x^2 + \frac{1}{3!}(a_0+2)x^3 + \cdots + \frac{1}{n!}(a_0+2)x^n + \dots.$$

The ratio test can again be used to verify that the radius of convergence is ∞.

Definition 1.7. A differential equation is called *linear* if it can be written in the following form.

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y'(x) + a_0(x)y(x) = r(x)$$

for some continuous functions $a_i(x)$ and $r(x)$.

7. Which of the differential equations from Example 1.2 are linear?

Solution: They all are linear!

8. Write down a new example of a non-linear differential equation and then solve it
Hint: For an easy one, do something like Example 1.6.

Solution: $y' = \frac{-1}{2}y^3$ is such an example. It is non-linear because y appears to the third power. However, it is still separable. Write $\frac{-2}{y^3}y' = 1$. Then $H(y) = \frac{1}{y^2}$ and $G(x) = x$. So $\frac{1}{y^2 = x + C}$. Thus $y = \frac{1}{\sqrt{x+C}}$ works as long as things make sense (ie, $x + C$ is positive). To check our solution, we notice that

$$y' = (-1)\left(\frac{s1}{\sqrt{x+C}}\right)^2 (1/2)\frac{1}{\sqrt{x+C}} = (-1/2)\left(\frac{1}{\sqrt{x+C}}\right)^3 = \frac{-1}{2}y^3.$$

9. Prove that the set of solutions to a linear differential equation form a vector space of functions under the assumption that $r(x)$ is the zero function.

Solution: Suppose that $f(x)$ and $g(x)$ are solutions to a linear differential equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$. Then

$$a_n(x)f^{(n)}(x) + \cdots + a_0(x)f(x) = 0 \text{ and}$$
$$a_n(x)g^{(n)}(x) + \cdots + a_0(x)g(x) = 0$$

Adding the two equations gives us

$$a_n(x)(f+g)^{(n)}(x) + \cdots + a_0(x)(f+g)(x) = 0$$

so that $f + g$ is also a solution. Now, for a constant $m \in \mathbb{R}$, we have

$$ma_n(x)f^{(n)}(x) + \cdots + ma_0(x)f(x) = 0$$

so that

$$a_n(x)(mf)^{(n)}(x) + \cdots + ma_0(x)(mf)(x) = 0$$

Thus mf is also a solution. So the set of solutions form a vector space of functions.

10. Find all constants A and B that make $y(x) = A\cos(x) + B\sin(x)$ a solution to (8) from Example 1.2.

Solution: Set $y(x) = A\cos(x) + B\sin(x)$. We get (for all x)

$$\begin{aligned} 0 = \quad & y'' && +y' && +2y \\ = \quad & (-A\cos(x) - B\sin(x)) && +(-A\sin(x) + B\cos(x)) && +(2A\cos(x) + 2B\sin(x)) \\ = \quad & (A+B)\cos(x) + (B-A)\sin(x) \end{aligned}$$

Since this must hold for $x = 0$, we see that $(A+B)\cos(0) + (B-A)\sin(0) = A + B = 0$. Since this must hold for $x = \pi/2$, we see that $(A+B)\cos(\pi/2) + (B-A)\sin(\pi/2) = B - A = 0$. Thus $A + B = 0$ and $A - B = 0$. Adding the equations gives $A = 0$, subtracting them gives $B = 0$. So we must have $y(x) = 0\cos(x) + 0\sin(x) = 0$ is the only solution of that form. (That wasn't very interesting...)

11. Find all constants A and B that make $y(x) = A\cos(x) + B\sin(x)$ a solution to (9) from Example 1.2.

Solution: Set $y(x) = A\cos(x) + B\sin(x)$ and then

$$\begin{aligned} y^{(3)}(x) = \quad & A\sin(x) - B\cos(x) \\ = \quad -y' = \quad & A\sin(x) - B\cos(x) \end{aligned}$$

Thus *every* pair of real numbers A and B yields a solution.

12. Find another solution to (9) from Example 1.2.
Hint: Think about one of the easiest functions you ever consider.

Solution: $y(x) = C$ works for any constant $C \in \mathbb{R}$.