WORKSHEET ON SIMILAR MATRICES, EIGENVECTORS AND CHARACTERISTIC POLYNOMIALS

MATH 186–1

Definition 0.1. Two 2×2 matrices A and B are called *similar* if there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that both A and B represent T but with respect to different bases.

In the homework due Friday, you will show the following.

Theorem 0.2. *Two matrices A and B are similar if and only if there exists an invertible matrix U such that $UAU^{-1} = B$.*

You may take this theorem as given for the purposes of this worksheet. Basically, you should view the matrix U as the matrix representing the linear transformation that sends one of the two bases to the other basis.

1. Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

Solution: Suppose that $A = UBU^{-1}$. Then

$$\begin{aligned}\det(A) = \det(UBU^{-1}) = \det(U)\cdot\det(B)\cdot\det(U^{-1}) &= \det(B)\cdot\det(U)\cdot\det(U^{-1})\\ &= \det(B)\cdot\det(UU^{-1}) = \det(B)\cdot(I) = \det(B).\end{aligned}$$

See problem #7 on Homework #7 for the first step. For the second step, recall the determinant is just a *number* and it doesn't matter what order we multiply numbers.

Geometric explanation: The determinant tells us by what factor area changes when using a linear transformation. This "factor" doesn't care about the particular basis you use (as long as you use the same basis for the domain and range).

2. Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Also give a geometric explanation.

Solution: Suppose that $A = UBU^{-1}$. Then the characteristic polynomial is equal to, $\det(\lambda I - A)$. I claim that $\lambda I - A$ is similar to $\lambda I - B$. To prove, this simply note that

$$\lambda I - A = \lambda(UIU^{-1}) - UBU^{-1}) = U(\lambda I - B)U^{-1}.$$

Now simply apply the previous problem and we are done.

Geometric explanation: At least in terms of the eigenvalues, these are numbers λ such that there exists a vector $\mathbf{v} \neq 0$ such that the associated linear transformation T satisfies $T(\mathbf{v}) = \lambda\mathbf{v}$. This has nothing to do with what particular basis you choose to represent T.

Definition 0.3. A matrix B is called *diagonal* if it is of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for some numbers a and b.

A matrix that is similar to a diagonal matrix is called *diagonalizable*.

3. Suppose that A is a matrix. Suppose that the linear transformation associated to A has two linearly independent eigenvectors. Prove that A is similar to a diagonal matrix.

Solution: Let T be the linear transformation associated with A. Consider the basis $\mathbf{u}, \mathbf{u}'$ of the two linearly indepent eigenvectors of A (and say they have associated eigenvalues λ, λ'). Then $T(\mathbf{u}) = \lambda\mathbf{u}$ and $T(\mathbf{u}') = \lambda'\mathbf{u}'$. If we represent T as a matrix with respect to the basis $\mathbf{u}, \mathbf{u}'$ then we obtain the matrix

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda' \end{bmatrix}.$$

This completes the proof because A is similar to this diagonal matrix by definition.

4. Prove that if A is a 2×2 matrix that has two distinct eigenvalues, then A is similar to a diagonal matrix.

Solution: Suppose A has two distinct eigenvalues λ, λ', we will show it has two linearly independent eigenvectors and then we will be done by the previous problem. Let $\mathbf{u} \neq 0$ be an eigenvector for λ and let $\mathbf{u}' \neq 0$ be an eigenvector for λ'.

Suppose that $\mathbf{u}, \mathbf{u}'$ are not linearly independent, thus they are scalar multiples of each other (notice they are non-zero). So there exists $c \neq 0$ such that $c\mathbf{u} = \mathbf{u}'$.

Then

$$\lambda'\mathbf{u}' = A\mathbf{u}' = A(c\mathbf{u}) = c(A\mathbf{u}) = c\lambda\mathbf{u} = \lambda(c\mathbf{u}) = \lambda(\mathbf{u}')$$

so that $(\lambda - \lambda')\mathbf{u}' = 0$. But then $\lambda = \lambda'$ which contradicts our initial assumption. Thus $\mathbf{u}, \mathbf{u}'$ is linearly independent and so we are done by the previous problem.

5. Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? (You showed the converse in the previous worksheet). Consider the example $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Is it true that matrices with equal characteristic polynomials are necessarily similar?

Solution: The characteristic polynomial of the given matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is $p(x) = x^2$. It has a double root (and only one eigenvalue, $\lambda = 0$). On the other hand, the only eigenvectors are vectors of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$.

- Thus matrices whose characteristic polynomials have a double root do not necessarily have two linear independent eigenvectors.

For the second statement, notice that the given matrix and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ both have the same characteristic polynomial, $p(x) = x^2$. But they are not similar. Here's why, the linear transformation assocated to the first matrix sends $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} x \\ 0 \end{bmatrix} \neq \mathbf{0}$. But the linear transformation associated to the second matrix sends every vector to $\mathbf{0}$. Thus the two matrices can't represent the same linear transformation. In particular,

- Matrices with equal characteristic polynomials are *NOT* necessarily similar

6. Suppose you are given a degree two polynomial $p(x) = x^2 + bx + c$. Construct a matrix whose characteristic polynomial is equal to $p(x)$.

Hint: Try making the first column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution: $\begin{bmatrix} 0 & -c \\ 1 & -b \end{bmatrix}$

Suppose that A is a matrix and $f(x) = a_n x^n + \cdots + a_1 x^1 + a_0 x^0$ is some polynomial. Consider the matrix $f(A) = a_n A^n + \cdots + a_1 A^1 + a_0 I$. Here:

(i) I is the identity matrix.
(ii) $A^n = A \cdot A \cdot \cdots \cdot A$, n-times.
(iii) For a scalar c and a matrix B, cB just scales every entry in the matrix by c.

7. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to a diagonal matrix and has characteristic polynomial $p(x)$. Prove that $p(A)$ is the zero matrix.

Solution: Write $D = UAU^{-1}$ for some invertible matrix U and some diagonal matrix $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Write notice that the characteristic polynomial for A and D are the same by problem (2) on this worksheet so that $p(x) = (x-a)(x-b)$. We will now prove that $p(A)$ is similar to $p(D)$.

$$\begin{aligned} Up(A)U^{-1} &= U(AA)U^{-1} + U(bA)U^{-1} + U(cI)U^{-1} \\ &= UAU^{-1}UAU^{-1} + bUAU^{-1} + cUIU^{-1} \\ &= DD + bD + cI \\ &= p(D) \end{aligned}$$

However,

$$p(D) = (D - aI)(D - bI) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $p(A)$ is similar to the zero matrix. But it is clear that the only matrix similar to the zero matrix is the zero matrix (the zero linear transformation is represented by the zero matrix no matter what basis you use). Thus $p(A) = 0$**8.** For any 2×2 matrix A with characteristic polynomial $p(x)$, prove that $p(A)$ is the zero matrix by brute force. This is called the Cayley-Hamilton theorem. It is also true for $n \times n$ matrices although brute force won't work in that case.

Solution: Set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $p(x) = (x-a)(x-d) - bc$. Thus

$$\begin{aligned} p(A) &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \right) - \begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\ &= \begin{bmatrix} 0 & b \\ c & d-a \end{bmatrix} \begin{bmatrix} a-d & b \\ c & 0 \end{bmatrix} - \begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\ &= \begin{bmatrix} bc & (a-d)c + c(d-a) \\ 0 & bc \end{bmatrix} - \begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$