WORKSHEET ON EIGENVALUES AND EIGENVECTORS

MATH 186–1

Definition 0.1. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.[1] A non-zero vector $\mathbf{v} \in \mathbb{R}^n$ is called a *eigenvector for T* if there exists a number λ such that $T(\mathbf{v}) = \lambda\mathbf{v}$. In this case, the number λ is called an *eigenvalue* for T.

1. Fix $\{\mathbf{u}, \mathbf{v}\}$ to be a basis for $\mathbb{R}^2$ and fix $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ to be a basis for $\mathbb{R}^3$. Given below are certain vectors and various linear transformations. In each case determine which vectors are eigenvectors and identify the associated eigenvalues.

(a) Set $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation represented by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Try the vectors, $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$.

$\mathbf{u}$ is an eigenvector with associated eigenvalue 2. $\mathbf{v}$ is an eigenvector with associated eigenvalue 3. The others are not eigenvectors.

(b) Set $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation represented by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Try the vectors $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$.

$\mathbf{u}+\mathbf{v}$ is an eigenvector with associated eigenvalue 1. $\mathbf{u}-\mathbf{v}$ is an eigenvector with associated eigenvalue -1. The others are not eigenvectors.

(c) Set $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation represented by the matrix $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Try the vectors $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$ and $\mathbf{u}+2\mathbf{v}$.

$\mathbf{u}+2\mathbf{v}$ is an eigenvector with associated eigenvalue 2. The others are not eigenvectors.

(d) Set $T : \mathbb{R}^3 \to \mathbb{R}^3$ to be the linear transformation represented by the matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$ where a, b are all distinct constants. Try the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x}+\mathbf{y}, 3\mathbf{x}-7\mathbf{y}$ and $\mathbf{x}+\mathbf{y}+\mathbf{z}$.

$\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}, 3\mathbf{x}-7\mathbf{y}$ are all eigenvectors associated with the eigenvalue a. $\mathbf{z}$ is an eigenvector with associated eigenvalue b. $\mathbf{x}+\mathbf{y}+\mathbf{z}$ is not an eigenvector.

[1]We will mostly be concerned with the case that $n = 2$

We'll now begin to develop a better method for identifying eigenvalues and eigenvectors than what we did on the previous page (guess and check). First fix some notation. We will use the letter I to denote the identity linear transformation. That is $I : \mathbb{R}^2 \to \mathbb{R}^2$ is the map defined by the formula $I(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}^2$.

2. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Let's suppose that a vector $\mathbf{w}$ is an eigenvector for T with associated eigenvalue λ. Prove that the new linear transformation $(\lambda \cdot I - T)$ is not injective. Here $(\lambda \cdot I - T)$ is defined by the rule $(\lambda \cdot I - T)(\mathbf{x}) = T(\mathbf{x}) - \lambda \cdot I(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$.

I'm going to assume that $\mathbf{w}$ is a non-zero vector. Eigenvectors are always assumed to be non-zero (I should have said this more clearly on the first page). Then

$$(\lambda \cdot I - T)(\mathbf{w}) = \lambda\mathbf{w} - T(\mathbf{w}) = \lambda\mathbf{w} - \lambda\mathbf{w} = \mathbf{0}.$$

But $(\lambda \cdot I - T)(\mathbf{0}) = \mathbf{0}$ also so $(\lambda \cdot I - T)$ sends two different vectors to zero and so it is not injective.

3. With notation as in problem #2, fix a basis $\{\mathbf{u}, \mathbf{v}\}$ for $\mathbb{R}^2$. Assume that T is represented by the matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$, write down a matrix representation of $(\lambda \cdot I - T)$. Finally write down the determinant of this matrix you constructed (note that this determinant is a polynomial in the variable λ, it is called the *characteristic polynomial of the matrix*).

The matrix representation of $(\lambda \cdot I - T)$ is $\begin{bmatrix} \lambda - e & -f \\ -g & \lambda - h \end{bmatrix}$. The determinant is

$$(\lambda - e)(\lambda - h) + fg$$

Remark 0.2. One can do something similar for 3×3 matrices. In particular, there is a determinant of such matrices and you can construct the characteristic polynomial in the same way. Versions of the results on the following pages also hold for 3×3 matrices.

4. Suppose that k is a real number. Show that that k is a root of the polynomial from problem 3. if and only if k is an eigenvalue for T.
Hint: In the homework you turned in yesterday, you showed that a if T was represented by a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then T is injective if and only if $ad - bc \neq 0$.

Fix a basis $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and fix a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ just as in the previous problem.

Suppose first that k is a root of the characteristic polynomial of this matrix. Then $\det(k \cdot I - T) = 0$ and in particular, $(k \cdot I - T)$ is not injective. But then there exists a non-zero $\mathbf{w}$ such that $(k\dot{I} - T)(\mathbf{w}) = \mathbf{0}$ or in other words $k\mathbf{w} = k\dot{I}(\mathbf{w}) = T(\mathbf{w})$ which proves that $\mathbf{w}$ is an eigenvector with associated eigenvalue k.

Conversely, suppose that k is an eigenvalue for the matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then it has an associated nonzero eigenvector $\mathbf{w}$. Thus $k\mathbf{w} = T(\mathbf{w})$ and it follows (reversing the steps from above) that $(k\dot{I} - T)(\mathbf{w}) = \mathbf{0}$. But then $\det(k\dot{I} - T) = (k - e)(k - h) + fg = 0$. Thus k is a root of the polynomial $(\lambda - e)(\lambda - h) + fg$.

5. Compute the eigenvalues of the linear transformations from problem #1(a),(b),(c). What's stopping you from computing the eigenvalues for the linear transformation corresponding to the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$? (Geometrically, remind yourself what this linear transformation does).

(a) The characteristic polynomial is $(\lambda - 2)(\lambda - 3)$ and so the roots (and thus eigenvalues) are 2 and 3.
(b) The characteristic polynomial is $\lambda^2 - 1$ and so the roots (and thus eigenvalues) are -1 and 1.
(c) The characteristic polynomial is $\lambda(\lambda - 1) - 2 = \lambda^2 - \lambda - 2$ and so the roots (and thus eigenvalues) are 2 and -1.
(d) The characteristic polynomial is $\lambda^2 + 1$. Thus polynomial doesn't have any roots! Geometrically, it corresponds to rotation by 90 degrees (and so geometrically, one would not expect any eigenvectors either).

6. Can a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ have more than 2 distinct eigenvalues? Justify your answer.

No, the eigenvalues of T are always the roots of a polynomial equation of degree 2. Such equations can have at most 2 roots (although sometimes they can also have 1 root or zero roots).

7. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a non-surjective linear transformation. Prove that $\lambda = 0$ is an eigenvalue for T.

Since T is non-surjective, it is non-injective. Thus $T(\mathbf{w}) = T(\mathbf{w}')$ for two distinct vectors $\mathbf{w}$ and $\mathbf{w}'$. Then $T(\mathbf{w} - \mathbf{w}') = T(\mathbf{w}) - T(\mathbf{w}') = \mathbf{0} = 0(\mathbf{w} - \mathbf{w}')$. In particular, 0 is an eigenvalue for the eigenvector $\mathbf{w} - \mathbf{w}'$.

8. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Further suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ are linearly independent eigenvectors of T but they have the same eigenvalue λ. Show that every vector in $\mathbb{R}^2$ is an eigenvector of T (associated to the same eigenvalue) and also that the characteristic polynomial of the matrix associated to T has a double-root at λ. What would it mean about T if $\lambda = 0$?

Fix any vector $\mathbf{w} \in \mathbb{R}^2$. Since $\mathbf{x}, \mathbf{y}$ are linearly independent, they are a basis and so we can write $\mathbf{w} = a\mathbf{x} + b\mathbf{y}$. But then

$$T(\mathbf{w}) = T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y}) = a\lambda\mathbf{x} + b\lambda\mathbf{y} = \lambda(a\mathbf{x} + b\mathbf{y}) = \lambda\mathbf{w}$$

as desired.

Now we show that the characteristic polynomial has a double root. We know that it has one root λ and so if we write the characteristic polynomial $z^2 + dz + e$ with the variable z (other letters already seem to be used), then $(z - \lambda)(z - ???) = z^2 + dz + e$ using polynomial long division. Let us use the variable γ instead of ???. Then γ must be an eigenvalue with associated eigenvector $\mathbf{w}' \neq \mathbf{0}$. But $\mathbf{w}' \in \mathbb{R}^2$ so $\mathbf{w}'$ is also an eigenvector associated to λ. In other words

$$\lambda\mathbf{w}' = T(\mathbf{w}') = \gamma\mathbf{w}'.$$

This implies that $\gamma = \lambda$.

Finally if $\lambda = 0$, then for any $\mathbf{w} = \mathbb{R}^2$, $T(\mathbf{w}) = 0\mathbf{w} = \mathbf{0}$. In particular, T is the linear transformation that sends all vectors to $\mathbf{0}$. It is represented by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ no matter what basis you use.

Now we turn to the question of finding the eigenvectors associated to a given eigenvalue. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation represented by a matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and that λ is an eigenvalue. To find the eigenvectors associated to λ, write

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now expand the left side of the equation and obtain equations (viewing x and y as variables). Find any pair of x and y that satisfy those equations and you have found an eigenvector. Let us do an explicit example:

Example 0.3. Suppose we are given the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. By the method described above, one can verify that the number 5 is an eigenvalue of the linear transformation associated to A. So we write

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}.$$

The right side of the equation is just $\begin{bmatrix} 5x \\ 5y \end{bmatrix}$ and the left side is $\begin{bmatrix} x + 2y \\ 4x + 3y \end{bmatrix}$. So we have the equations

$$x + 2y = 5x$$
$$5y = 4x + 3y$$

Which reduces (in either case) to $y = 2x$. Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector (as is $\begin{bmatrix} -7 \\ -14 \end{bmatrix}$).

9. Using this method, find the eigenvectors associated to the matrices from problem #1(a)(b)(c). Also, find the eigenvalues and eigenvectors associated to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. This last one is fairly messy.

(a) All scalar multiples of $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are the eigenvectors for the eigenvalue 2. All scalar multiples of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the eigenvectors for the eigenvalue 3.

(b) All scalar multiples of $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the eigenvectors for the eigenvalue 1. All scalar multiples of $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the eigenvectors for the eigenvalue -1.

(c) All scalar multiples of $\mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are the eigenvectors for the eigenvalue 2. All scalar multiples of $-\mathbf{u} + \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are the eigenvectors for the eigenvalue -1.

(d) All scalar multiples of $\mathbf{u} + \frac{3+\sqrt{33}}{4}\mathbf{v} = \begin{bmatrix} 1 \\ \left(\frac{3+\sqrt{33}}{4}\right) \end{bmatrix}$ are the eigenvectors for the eigenvalue $\frac{5+\sqrt{33}}{2}$. All scalar multiples of $\mathbf{u} + \frac{3-\sqrt{33}}{4}\mathbf{v} = \begin{bmatrix} 1 \\ \left(\frac{3-\sqrt{33}}{4}\right) \end{bmatrix}$ are the eigenvectors for the eigenvalue $\frac{5-\sqrt{33}}{2}$.