PRACTICE FOR EXAM #1

1. Write down the Taylor series for the following functions centered at a.

(a) $f(x) = e^{(x^2)}$ centered at $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Note the uniqueness of power series (see the last page of problems) implies that this must be the Taylor series.

(b) $g(x) = \sin(\pi x)$ centered at $a = 1$.

$$g(x) = -\pi(x-1) + \pi^3(x-1)^3/3! - \pi^5(x-1)^5/5! + \pi^7(x-1)^7/7! + \dots$$

(c) $h(x) = \frac{e^x - e^{-x}}{2}$ centered at $a = 0$.

$$h(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(1 + (-1)^{n+1})x^n}{n!}$$

(d) $f(x) = x^3$ centered at $a = 1$.

$$1 + \frac{3(x-1)^1}{1!} + \frac{(6)(x-1)^2}{2!} + \frac{(6)(x-1)^3}{3!} = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3 + 0(x-1)^4 + 0(x-1)^5 + \dots$$

(e) $r(x) = \frac{2}{4-x}$ centered at $a = 0$.

Note that $2r(4x) = \frac{4}{4-4x} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$. Therefore

$$r(x) = \frac{1}{2} + (\frac{1}{2})(\frac{x}{4}) + (\frac{1}{2})(\frac{x^2}{4^2}) + (\frac{1}{2})(\frac{x^3}{4^3}) + \dots$$

2. Determine which of the following series are convergent. Use the comparison tests, ratio test, integral test, and Leibnez's theorem.
(a) $\sum_{n=4}^{\infty} \frac{(-1)^n}{\sqrt{n-2}}$

Converges. This follows immediately from Leibnez's theorem since $\frac{1}{\sqrt{n-2}}$ is a decreasing sequence of positive numbers which converges to zero.

(b) $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^n}$

Converges. To see this it is sufficient to show that the partial sum $\sum_{n=9}^{\infty} \frac{1}{(\ln(n))^n}$ converges. But then we apply the comparison test with the series $\sum_{n=9}^{\infty} \frac{1}{2^n}$ (note that $\ln(n) > 2$ for $n \geq 9$).

(c) $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

Diverges. We use the ratio test:

$$\frac{3^{n+1}(n+1)!/(n+1)^{(n+1)}}{3^n n!/n^n} = \frac{3n^n}{(n+1)^n} = \left(\frac{3}{(1+\frac{1}{n})^n}\right)$$

As $n \to \infty$ this goes to $\frac{3}{e} > \frac{3}{3} = 1$ which shows it diverges by the ratio test.

(d) $\sum_{n=2}^{\infty} \frac{1}{n\cos(\pi n)}$

Converges, the series is exactly the same as $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n}$ which converges by Leibnez's theorem.

(e) $\sum_{n=1}^{\infty} \frac{n!}{n^{(n!)}}$

Converges, use the comparison test. Note $\frac{n!}{n^{(n!)}} < \frac{n!}{n^n}$ and the series of the latter converges by work in previous assignments.

3. Show that the following sequences $\{a_n\}$ are convergent or non-convergent.

(a) $a_n = \sqrt{\ln(n+1)}$.

Non-convergent, in fact the sequence is not even bounded. Fix $K > 0$. Then set $N = \lceil e^{K^2} \rceil$. Note that for $n > N$, $a_n > a_N = \sqrt{\ln(\lceil e^{K^2} \rceil + 1)} > \sqrt{\ln(e^{K^2})} = \sqrt{K^2} = K$.

(b) Set $a_1 = 0$ and $a_2 = 3$. In general, for $n \geq 3$, use the recursive definition $a_n = (2a_{n-1} + a_{n-2})/3$.

Convergent, we will in fact show that the sequence is Cauchy. First consider

$$|a_{n+1} - a_n| = |(2a_n + a_{n-1})/3 - a_n| = \frac{1}{3}|a_n - a_{n-1}|.$$

Therefore in general we know that $|a_{n+1} - a_n| = \frac{1}{3}^{n-1}|a_2 - a_1| = \frac{1}{3}^{n-1}(3) = \frac{1}{3}^{n-2}$. Choose $\varepsilon > 0$ and choose $N > 0$ such that $\frac{1}{3}^{N-2} < \varepsilon$. Suppose that $n, m > N$. Notice that a_n and a_m are in between a_N and a_{N+1} by construction, so that $|a_n - a_m| < |a_N - a_{N+1}| = \frac{1}{3}^{N-2} < \varepsilon$ as desired.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded non-decreasing function. Define $a_1 = f(0)$ and define $a_n = f(a_{n-1})$ recursively.

Convergent, but we must analyze several cases.

Case 1 $a_1 = 0$. In this case $a_2 = 0$, $a_3 = 0$, etc... So that a_n is a constant and thus convergent sequence.

Case 2 $a_1 > 0$. In this case $a_2 = f(a_1) \geq f(0) = a_1$. And in general, by induction (assuming $a_n \geq a_{n-1}$), we know that $a_{n+1} = f(a_n) \geq f(a_{n-1}) = a_n$. Therefore a_n is a non-decreasing sequence of numbers which is bounded above. It therefore converges.

Case 3 $a_1 < 0$. In this case, $a_2 = f(a_1) \leq f(0) = a_1$. And in general, by induction (assuming $a_n \leq a_{n-1}$), we know that $a_{n+1} = f(a_n) \geq f(a_{n-1}) = a_n$. Therefore a_n is a non-increasing sequence of numbers which is bounded below. It therefore converges.

(d) Let $\{b_n\} \to b$ be a convergent sequence and suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Define $a_n = f(b_n)$.

Convergent, we will show directly that it converges to $f(b)$. Choose $\varepsilon > 0$. Because f is continuous, there exists δ bigger than zero such that if $|x - b| < \delta$ then $|f(x) - f(b)| < \varepsilon$. Because $\{b_n\}$ converges to b there exists $N > 0$ such that if $n > N$ then $|b_n - b| < \delta$. Then notice that if $n > N$, we know that $|b_n - b| < \delta$ so that $|f(b_n) - f(b)| < \varepsilon$ as desired.

(e) Let b_n denote the number of prime numbers less than or equal to n (for example, $b_6 = 3$, because $2, 3, 5$ are prime and all are less than or equal to 6). Then define $a_n = \sum_{n=3}^{\infty} \frac{1}{b_n}$.

It diverges by the comparison test since $\frac{1}{b_n} \geq \frac{1}{n}$. (ie, there are more integers less than n than *prime* integers less than n).

4. Give a direct ϵ-N proof of the following fact without looking it up in the book: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that $\{a_n\}$ is a sequence which converges to a. Show that $\{f(a_n)\}$ converges to $f(a)$.

This was already given in 3(d).

5. Let $f_n : A \to \mathbb{R}$ be a sequence of functions. Show that the infinite series of functions $\sum_{k=1}^{\infty} f_k$ converges uniformly on A if and only if for every $\epsilon > 0$, there exists an integer $N > 0$ such for any pair of integers m and n satisfying $m > n > N$, then

$$|\sum_{k=n+1}^{m} f_k(x)| < \epsilon$$

for all $x \in A$.

The series $\sum_{k=1}^{\infty} f_k$ converges if and only if the sequence of partial sums $s_n = \sum_{k=1}^{n} f_k$ converges if and only if the sequence of partial sums $\{s_n\}$ is Cauchy. But the sequence of partial sums is Cauchy if and only for every $\epsilon > 0$ there exists an integer $N > 0$ such that if $m, n > N$, then $|s_n - s_m| < \epsilon$. Now, if $m > n$, then $|s_n - s_m| = |\sum_{k=n+1}^{m} f_k(x)|$. On the other hand, if $n \geq m$, we can reverse the roles of m and n and get the same conclusion.

6. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for all $x \in [-c, c]$ for some $c > 0$ (in particular, both $\sum_{n=0}^{\infty} a_n c^n$ and $\sum_{n=0}^{\infty} b_n c^n$ converge). Show that $a_n = b_n$ for all n.

Note that f is thus differentiable (and its derivatives themselves are differentiable etc) for $x \in (-c, c)$ based on a Theorem we proved in class (and in the book). But notice that $f^{(n)}(0) = n!a_n = n!b_n$ and so we conclude that $a_n = b_n$ as desired.