WORKSHEET ON LINEAR TRANSFORMATIONS

MATH 186–1

Definition 0.1. Given a function T (which takes vectors as input, and outputs vectors), we say that T is a *linear transformation* if the following two properties hold.

(#1) For any two vectors $\mathbf{v}$ and $\mathbf{v}'$, we always have $T(\mathbf{v}+\mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}')$. In other words, adding the vectors and then using the function gives the same result as using the function, then adding the two outputs.

(#2) For any $\mathbf{v}$ and any real number c, we have $T(c\mathbf{v}) = cT(\mathbf{v})$. In other words, if you scale the vector and then apply the function, you get the same result as if you'd applied the function and then scaled the output.

1. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation ($\mathbb{R}^2$ we view as the set of vectors in the plane, and it includes the zero vector). Fix $\mathbf{u}$ and $\mathbf{v}$ to be a basis. For all other vectors we write $\mathbf{x} = a_x\mathbf{u} + b_x\mathbf{v}$ because $\mathbf{u}, \mathbf{v}$ are a basis.[1] Show that $T(\mathbf{x})$, for any $\mathbf{x}$, is completely determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

2. Fix a basis $\mathbf{u}$, $\mathbf{v}$ for $\mathbb{R}^2$. Show that for every other pair of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, that there exists a linear transformation T such that $T(\mathbf{u}) = \mathbf{x}$ and $T(\mathbf{v}) = \mathbf{y}$.

[1]Other common expressions for $\mathbf{x}$ include $(a_{\mathbf{x}}, b_{\mathbf{x}}) = \begin{bmatrix} a_{\mathbf{x}} \\ b_{\mathbf{x}} \end{bmatrix}$.

3. Write down matrix representations of the following linear transformations also explain as well as you can what this linear transformation does geometrically. Fix a basis $\mathbf{u}, \mathbf{v}$ for $\mathbb{R}^2$ and a basis $\mathbf{x}, \mathbf{y}, \mathbf{z}$ for $\mathbb{R}^3$.

(a) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined in the following way. $T(\mathbf{u}) = \mathbf{v}$ and $T(\mathbf{v}) = \mathbf{u}$.

(b) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined in the following way. $T(\mathbf{u}) = \mathbf{v}$ and $T(\mathbf{v}) = -\mathbf{u}$.

(c) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined in the following way. $T(\mathbf{u}) = \mathbf{u}$ and $T(\mathbf{v}) = \mathbf{0}$.

(d) The map $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined in the following way. $T(\mathbf{u}) = \mathbf{x}$ and $T(\mathbf{v}) = \mathbf{y} + \mathbf{z}$.

(e) The map $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined in the following way. $T(\mathbf{x}) = \mathbf{u}$ and $T(\mathbf{y}) = \mathbf{v}$ and $T(\mathbf{z}) = \mathbf{0}$.